Combinatorics of the dispersionless Toda hierarchy

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- Catalan numbers
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- The dispersionless Toda hierarchy
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Catalan numbers

Consider a genus 0 curve (*sphere with two marked points*):

$$\lambda = p + \frac{1}{p} \quad (\text{or} \quad \lambda p = p^2 + 1)$$

Expand p in terms of a large λ with $p \to \lambda$ as $\lambda \to \infty$, i.e.

$$p = \lambda - \sum_{n=0}^{\infty} \frac{C_n}{\lambda^{2n+1}}$$

The coefficients C_n are the Catalan numbers:

$$C_n = -\oint_{\lambda=\infty} \frac{d\lambda}{2\pi i}\, p(\lambda)\lambda^{2n} = -\oint_{p=\infty} \frac{dp}{2\pi i} \left(p - \frac{1}{p}\right)\left(p + \frac{1}{p}\right)^{2n}$$

Catalan numbers

Explicitly the n-th Catalan number is given by

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n} \quad n \geq 0.$$

The Catalan numbers satisfy the recurrence relation (from the curve, i.e. the generating function of C_n),

$$C_0 = 1, \quad C_{n+1} = \sum_{i+j=n} C_i\, C_j.$$

Examples:

$$C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14, C_5 = 42, \ldots$$

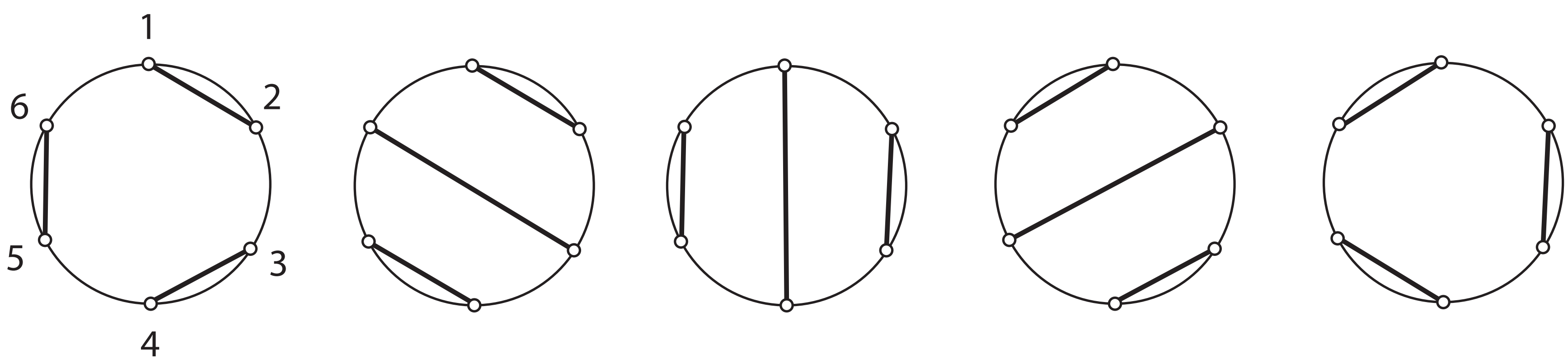
(*Note that* C_n *=odd, iff* $n = 2^k - 1$.)

Catalan numbers

"Enumerative Combinatorics" (*Stanley*) contains 66 different interpretations of the Catalan number. The most relevant one to our study is:

"*C_n gives the number of ways to make n non-crossing chords joining pairs of $2n$ points on a circle.*"

Example: $n = 3, C_3 = 5$,



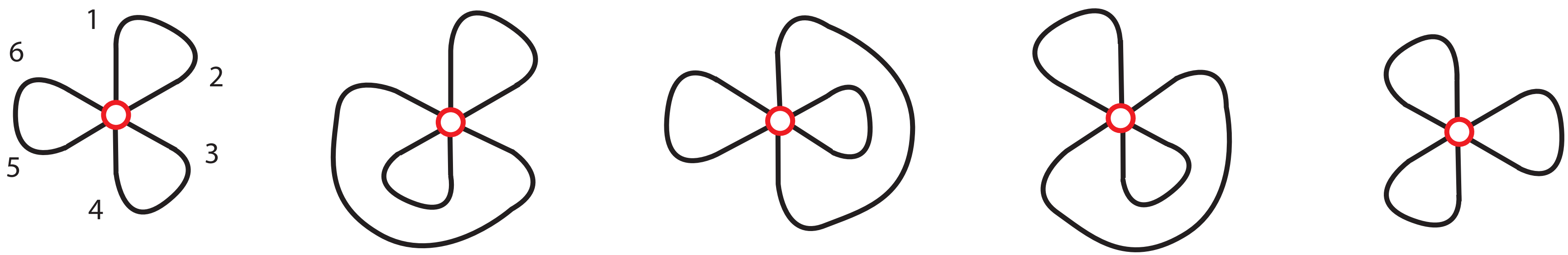
Proof: Recall that C_n satisfy the recurrence relation,

$$C_{n+1} = \sum_{i+j=n} C_i C_j .$$

Catalan numbers

This also gives the number of ways to make n non-crossing *ordered* ribbons for one-vertex of degree $2n$ on a sphere.

Example: $n = 3, C_3 = 5$,

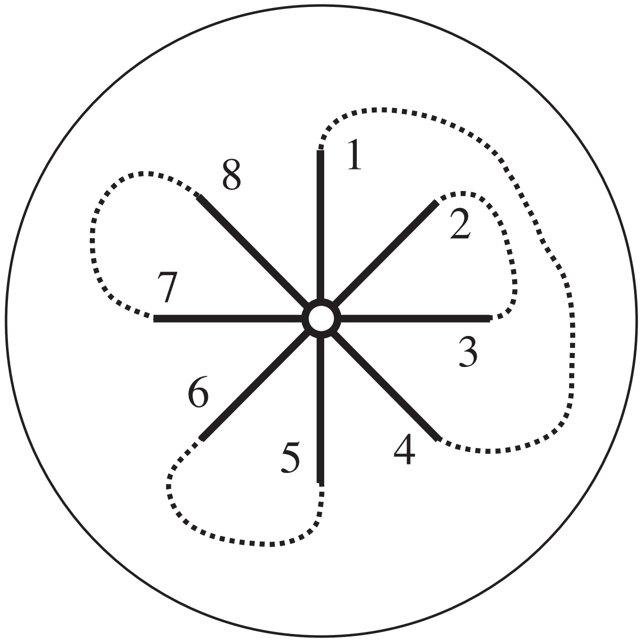


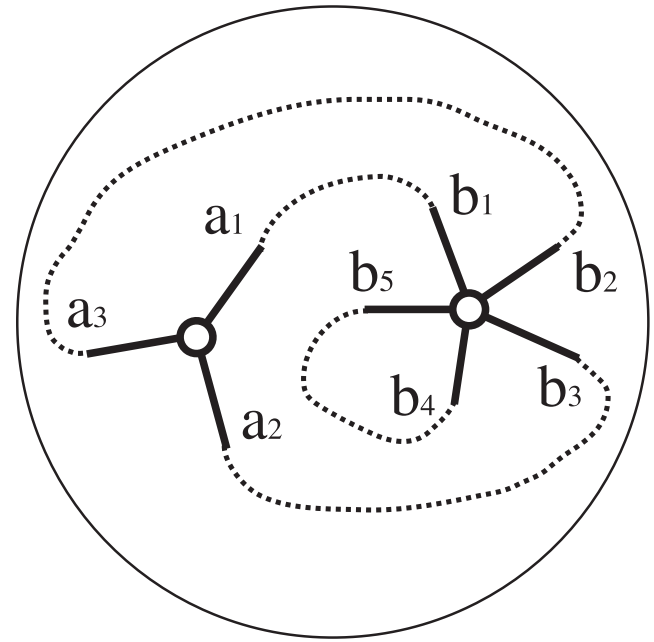
Note that if the degree is odd, then the number of ribbon graphs is zero (we do not count incomplete graph). This problem is called one-vertex problem, and we here consider two-vertex problem:

"*Find the number of ways to make connected ribbon graph with two vertices of degrees n and m; the number is denoted by F_{mn}.*"

Catalan numbers

One- and Two-vertex problems on a sphere:





(1). C_n *gives the solution of the one-vertex problem with a vertex of degree* $2n$*.*

(2). F_{mn} *gives the solution of the two-vertwx problem with vertices of degrees* m *and* n*. (We give an explicit form of* F_{mn}*.)*

Gaussian unitary ensemble (GUE)

The partition function of the GUE is defined by

$$Z_n(V_0;\mathbf{t}) = \int_{\mathbb{R}^n} d\vec{\lambda}\ \prod_{i<j}(\lambda_i-\lambda_j)^2 \exp\left[-\sum_{j=1}^{n} V_0(\lambda_j) + \sum_{k=1}^{\infty} t_k \lambda_j^k\right]$$

Introduce the slow scales $\mathbf{T} = \mathbf{t}/N = (T_1, T_2, \ldots)$ and $T_0 = n/N$, and consider the limit $N \to \infty$. Then we have:

Theorem [*Bessis et al. (1986)*] *With* $V_0(\lambda) = \frac{N}{2}\lambda^2$*, the logarithm of the partition function has an asymptotic expansion of the form,*

$$\log\left[Z_N\left(\frac{N}{2}\lambda^2; N\mathbf{T}\right) \Big/ Z_N\left(\frac{N}{2}\lambda^2; \mathbf{0}\right)\right] = \sum_{g\geq 0} e_g(\mathbf{T}) N^{2-2g}.$$

Gaussian unitary ensemble (GUE)

Here the coefficients $e_g(\mathbf{T})$ are given by

$$e_g(\mathbf{T}) = \sum_{0 \le j_1, j_2, \dots} \kappa_g(j_1, j_2, \dots) \frac{T_1^{j_1} T_2^{j_2} \cdots}{j_1! j_2! \cdots} = \sum_{\mathbf{j}} \kappa_g(\mathbf{j}) \frac{\mathbf{T}^{\mathbf{j}}}{\mathbf{j}!} .$$

The coefficient $\kappa_g(\mathbf{j})$ gives the number of the connected ribbon graphs with j_k labeled vertices of degree k for $k = 1, 2, \dots$ on a compact surface of genus g.

In particular, we have

$$e_0(\mathbf{T}) = \lim_{N \to \infty} \frac{1}{N^2} \log \left[Z_N \left(\frac{N}{2} \lambda^2 ; N\mathbf{T} \right) / Z_N \left(\frac{N}{2} \lambda^2 ; \mathbf{0} \right) \right] .$$

Gaussian unitary ensemble (GUE)

Example: In this limit, we have

$$\frac{\partial e_0}{\partial T_n}(\mathbf{T})\Big|_{\mathbf{T}=0} = \kappa_0(0,\ldots,0,\overset{n}{1},0,\ldots) = \begin{cases} C_k, & \text{if } n = 2k \\ 0, & \text{otherwise} \end{cases}$$

Also the quantity with $mn \neq 0$,

$$\frac{\partial^2 e_0}{\partial T_m \partial T_n}(\mathbf{T})\Big|_{\mathbf{T}=0} = \kappa_0(0,\ldots,\overset{m}{1},\ldots,\overset{n}{1},\ldots)$$

gives the number of connected ribbon graphs with two vertices of degrees m and n (i.e. *Two-vertex problem*).

Find an explicit formula for this quantity, i.e. F_{mn}!!!

The Toda lattice hierarchy

The Toda lattice hierarchy is defined by

$$\frac{\partial L}{\partial t_n} = [L, A_n], \quad \text{with} \quad L := \begin{pmatrix} b_1 & 1 & & & \\ a_1 & b_2 & 1 & & \\ & a_2 & b_3 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where $A_n := [L^n]_{<0}$ is the lower triangular part of L^n. In terms of the τ-functions, (a_k, b_k) are given by

$$a_k = \frac{\partial^2}{\partial t_1^2} \ln \tau_k = \frac{\tau_{k+1}\tau_{k-1}}{\tau_k^2}, \qquad b_k = \frac{\partial}{\partial t_1} \ln \frac{\tau_k}{\tau_{k-1}},$$

with $\tau_0 = 1$.

The Toda lattice hierarchy

The Toda hierarchy in Hirota bilinear form:

$$D_1^2 \tau_n \cdot \tau_n = 2\tau_{n+1}\tau_{n-1}$$

$$(D_k - h_k(\tilde{\mathbf{D}}))\tau_{n+1} \cdot \tau_n = 0.$$

where $\tilde{\mathbf{D}} = (D_1, \frac{1}{2}D_2, \ldots)$ with the usual Hirota derivative,

$$D_k f \cdot g = \lim_{s \to 0} \frac{d}{ds} f(t_k + s) g(t_k - s).$$

and $h_k(\mathbf{x})$ is the elementary symmetric polynomial,

$$\exp\left(\sum_{n=1}^{\infty} x_k z^k\right) = \sum_{k=0}^{\infty} h_k(\mathbf{x}) z^k.$$

The Toda lattice hierarchy

The first equation of the Toda hierarchy implies that τ_n can be written in the Hankel determinant form,

$$\tau_n = \begin{vmatrix} \tau_1 & \tau_1' & \cdots & \tau_1^{(n-1)} \\ \tau_1' & \tau_1'' & \cdots & \tau_1^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_1^{(n-1)} & \tau_1^{(n)} & \cdots & \tau_1^{(2n-2)} \end{vmatrix}.$$

The second equation for $n = 0$ implies that τ_1 is a solution of the linear PDE's,

$$\frac{\partial \tau_1}{\partial t_k} = h_k(\tilde{\mathbf{D}})\tau_1 = \frac{\partial^k \tau_1}{\partial t_1^k}.$$

The Toda lattice hierarchy

Writing the solution of this PDE in the form,

$$\tau_1 = \int_{\mathbb{R}} e^{\theta(\mathbf{t};\lambda)} \rho(\lambda)\, d\lambda, \quad \text{with } \theta(\mathbf{t};\lambda) = \sum_{k=1}^{\infty} \lambda^k t_k,$$

one can show that the partition functions $Z_n(V_0;\mathbf{t})$ are related to the τ-functions with $\rho(\lambda) = e^{-V_0(\lambda)}$,

$$\tau_n(\mathbf{t}) = \frac{1}{n!} Z_n\,(V_0;\mathbf{t}).$$

In particular, we consider the case with $V_0 = \frac{N}{2}\lambda^2$, i.e.

$$\tau_n(\mathbf{t};N) := \frac{1}{n!} Z_n \left(\frac{N}{2}\lambda^2;\mathbf{t} \right).$$

Large N limit of GUE

With the slow variables $T_0 = n/N$ and $\mathbf{T} = \mathbf{t}/N$, we compute the limit

$$F(T_0, \mathbf{T}) := \lim_{N\to\infty} \frac{1}{N^2} \log \left[\frac{1}{n!} Z_n \left(\frac{N}{2}\lambda^2; N\mathbf{T} \right) \right].$$

The $F(T_0, \mathbf{T})$ is called the free energy for a topological field theory (TFT) related to $\mathbb{C}P^1$ σ-model. Using Mehta's formula $Z_n(\lambda^2; 0) = (2\pi)^{n/2} 2^{-n^2/2} \prod_{j=1}^{n} j!$ with Stirlings' approximation $\log(n!) = \mathcal{O}(n \log n)$, we have

$$F(T_0, \mathbf{T}) = T_0^2\, e_0(\hat{\mathbf{T}}) + \frac{T_0^2}{2} \left(\log T_0 - \frac{3}{2} \right),$$

where $\hat{\mathbf{T}} = (\hat{T}_1, \hat{T}_2, \ldots)$ with $\hat{T}_j := T_0^{j/2-1} T_j$ (Penner scaling).

Large N limit of GUE

In the TFT, the second derivatives of the free energy play the essential role, and those are called two-point functions:

$$F_{mn} := \frac{\partial^2 F}{\partial T_m \partial T_n}.$$

In particular, Theorem [BIZ] implies that $F_{mn}(1, \mathbf{0})$ for $mn \neq 0$ represents the number of connected ribbon graphs with two vertices of degrees m and n on a sphere, that is, the solution of the two-vertex problem,

$$F_{mn}(1, \mathbf{0}) := \frac{\partial^2 F}{\partial T_m \partial T_n}(1, \mathbf{0}) = \kappa_0(0, \ldots, \overset{m}{1}, \ldots, \overset{n}{1}, \ldots), \quad nm \neq 0.$$

Large N limit of GUE

In the case of $m = 0$ and $n = 2k \neq 0$, we have

$$F_{0,2k}(1, \mathbf{0}) = (k+1)\kappa_0(0, \ldots, \overset{2k}{1}, \ldots),$$

This corresponds to counting the number of connected ribbon graphs with a vertex of degree $2k$ and a marked face on a sphere, which is actually given by

$$F_{0,2k}(1, \mathbf{0}) = (k+1)\, C_k.$$

Here $k+1$ represents the number of connected regions bounded by the ribbons.

The dispersionless Toda hierarchy

The free energy $F(T_0, \mathbf{T})$ is now defined in terms of the τ-function,

$$F(T_0, \mathbf{T}) = \lim_{N\to\infty} \frac{1}{N^2} \log \tau_n(N\mathbf{T}; N).$$

The Toda lattice has the limits,

$$\frac{\partial^2}{\partial t_1^2} \log \tau_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} \quad \longrightarrow \quad F_{11} = e^{F_{00}}$$

$$(D_k - h_k(\tilde{\mathbf{D}}))\tau_{n+1} \cdot \tau_n = 0 \quad \longrightarrow \quad F_{0k} = h_k(\mathbf{Z})$$

where $\mathbf{Z} = (Z_1, Z_2, \ldots)$ is defined by

$$Z_1 = F_{01}. \quad Z_n = \frac{F_{0n}}{n} + \sum_{k+l=n} \frac{F_{kl}}{kl}.$$

The dispersionless Toda hierarchy

The spectral problem $L\phi = \lambda\phi$ gives a plane curve: That is, for $a_{n-1}\phi_{n-1} + b_n\phi_n + \phi_{n+1} = \lambda\phi_n$, we write

$$\phi_n = e^{NS_n} \qquad (\text{WKB form}).$$

which represents a fast oscillation in the phase. Then writing

$$\frac{\phi_{n+1}}{\phi_n} = e^{\ln\phi_{n+1} - \ln\phi_n} = e^{N(S_{n+1} - S_n)},$$

we define

$$p := \lim_{N\to\infty} e^{N(S_{n+1} - S_n)} = \exp\left(\frac{\partial S}{\partial T_0}\right).$$

This is a quasi-momentum in the semi-classical limit.

The dispersionless Toda hierarchy

Then in the limit $N \to \infty$, the spectral problem then gives the curve,

$$\lambda = p + F_{01} + \frac{F_{11}}{p}.$$

Here note that $a_n \to F_{11} = e^{F_{00}},\ b_n \to F_{01}$.

Remark: The S in the momentum p is given by

$$S = \sum_{k=1}^{\infty} \lambda^k T_k + T_0 \ln \lambda - D(\lambda) F_0,$$

with $D(\lambda)$ defined by

$$D(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n\lambda^n} \frac{\partial}{\partial T_n}.$$

The dispersionless Toda hierarchy

The dispersionless Toda (dToda) hierarchy can be defined in the form,

$$\begin{cases} 1 - \dfrac{e^{F_{00}}}{p(\lambda)p(\mu)} = e^{-D(\lambda)D(\mu)F} \\ \lambda = p(\lambda) + F_{01} + \dfrac{e^{F_{00}}}{p(\lambda)} \quad \text{with} \quad p(\lambda) = \lambda e^{-D(\lambda)F_0} \, . \end{cases}$$

The second equation defines a plane curve (dToda curve), and the first equation gives its integrable deformation. We can also derive the equation without F_{00} term,

$$\frac{p(\lambda) - p(\mu)}{\lambda - \mu} = e^{D(\lambda)D(\mu)F}.$$

This is the dispersionless KP hierarchy, i.e. dToda $\subset$ dKP.

The dispersionless Toda hierarchy

Remark that the dToda hierarchy expressed by F_{mn} is completely determined by F_{01} and F_{00}. For example,

$$D(\lambda)F_0 = \log\frac{\lambda}{p(\lambda)} = \log\frac{2\lambda}{\lambda - F_{01} + \sqrt{(\lambda - F_{01})^2 - 4F_{11}}}.$$

To find the formula F_{mn}, we use the Faber polynomials for the dToda curve:

Proposition: The Faber polynomial $\Phi_n(p)$ is expressed by

$$\Phi_n(p) := [\lambda(p)^n]_+ = \lambda^n - D(\lambda)F_n = \lambda^n - \sum_{m=1}^{\infty}\frac{F_{mn}}{m\lambda^m}.$$

where $[\lambda(p)^n]_+$ is the polynomial part of $\lambda(p)^n$ in p.

The dispersionless Toda hierarchy

With those equations for F_{mn}, one can find the explicit formula for F_{mn} at $T_0 = 1, \mathbf{T} = \mathbf{0}$:

Theorem [*K-Pierce (2009)*]: With $F_{01} = F_{00} = 0$ (i.e. $F_{11} = 1$), we have

$$\begin{cases} F_{0,2k} &= (k+1)C_k, \\ F_{2j+1,2k+1} &= (2j+1)(2k+1)\frac{(j+1)(k+1)}{j+k+1}C_jC_k, \\ F_{2j,2k} &= jk\frac{(j+1)(k+1)}{j+k}C_jC_k, \\ F_{mn} &= 0, \qquad \text{otherwise}, \end{cases}$$

The F_{mn} gives the solution of the two-vertex problem.