On the Factorizations of Rational Matrix Functions with Applications to Integrable Systems and Discrete Painlevé Equations

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Outline

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- applications to discrete Painlevé equations.

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- Hamiltonian $\qquad$ Lagrangian

  $H_{n,p} \sim \operatorname{Tr}\mathbf{L}^{n+1}(p)$ $\qquad$ **?**

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$$\mathbf{L}(z) = \mathbf{L}_1(z)\mathbf{L}_2(z) \mapsto \tilde{\mathbf{L}}(z) = \mathbf{L}_2(z)\mathbf{L}_1(z), \qquad \mathbf{R}(z) = \mathbf{L}_2(z)$$

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- Such transformations sometimes reduce to discrete Painlevé equations

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- The divisor $(\mathbf{L}(z)) = (\det \mathbf{L}(z))$ *is simple as well*,
$$(\det \mathbf{L}(z)) = \sum_{k=1}^{n} (z_k - \zeta_k), \qquad \det \mathbf{L}(z) = \rho_1 \cdots \rho_m \prod_{k=1}^{n} \frac{z - \zeta_k}{z - z_k}.$$
This is the **rank-one** condition on the residues:
$$\mathbf{L}_k = \operatorname{res}_{z_k} \mathbf{L}(z) = \mathbf{a}_k \mathbf{b}_k^\dagger = \alpha_k [\mathbf{a}_k][\mathbf{b}_k^\dagger],$$
$$\mathbf{M}_k = -\operatorname{res}_{\zeta_k} \mathbf{M}(z) = \mathbf{c}_k \mathbf{d}_k^\dagger = \beta_k [\mathbf{c}_k][\mathbf{d}_k^\dagger].$$

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$$\mathbf{M}(z) = \mathbf{M}_0 - \sum_{k=1}^{n} \frac{\mathbf{M}_k}{z - \zeta_k}, \qquad \mathbf{M}_0 = \operatorname{diag}\{\frac{1}{\rho_1}, \ldots, \frac{1}{\rho_m}\} \qquad \mathbf{M}_k = \mathbf{c}_k \mathbf{d}_k^{\dagger}$$

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Note that

$$\mathbf{d}_k^\dagger \mathbf{L}(\zeta_k) = \mathbf{0}, \quad \mathbf{L}(\zeta_k)\mathbf{c}_k = \mathbf{0}, \quad \mathbf{b}_k^\dagger \mathbf{M}(z_k) = \mathbf{0}, \quad \mathbf{M}(z_k)\mathbf{a}_k = \mathbf{0}.$$

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The space of such $\mathbf{L}(z)$ with the fixed $\mathcal{D} = \sum_i z_i - \sum_i \zeta_i$ is $\mathcal{M}_r^{\mathcal{D}}$.

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Question

What are good coordinate systems on the space $\mathcal{M}_r^{\mathcal{D}}$?

Lemma

Generically, the collection $\{\mathbf{a}_k, \mathbf{d}_k^\dagger\}_{k=1}^n$ *(or the collection* $\{\mathbf{c}_k, \mathbf{b}_k^\dagger\}_{k=1}^n$*) gives coordinates on the space* $\mathcal{M}_r^{\mathcal{D}}$*.*

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Consider the equations $\mathbf{M}(z_k)\mathbf{a}_k = \mathbf{0}$ and $\mathbf{d}_i^\dagger \mathbf{L}(\zeta_i) = \mathbf{0}$:

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$$\mathbf{c}_i = \mathbf{L}_0^{-1}\mathbf{a}_k \left[\frac{\mathbf{d}_i^\dagger \mathbf{a}_k}{z_k - \zeta_i}\right]^{-1}, \quad \mathbf{b}_k^\dagger = \left[\frac{\mathbf{d}_i^\dagger \mathbf{a}_k}{z_k - \zeta_i}\right]^{-1} \mathbf{d}_i^\dagger \mathbf{L}_0.$$

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Unfortunately, Lagrangian coordinates seem to be a mix of vectors from these two collections.

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Elementary Divisors (building blocks)

$$\mathbf{B}(z) = \mathbf{I} + \frac{z_0 - \zeta_0}{z - z_0} \frac{\mathbf{p}\mathbf{q}^\dagger}{\mathbf{q}^\dagger \mathbf{p}} \qquad \mathbf{B}(z)^{-1} = \mathbf{I} + \frac{\zeta_0 - z_0}{z - \zeta_0} \frac{\mathbf{p}\mathbf{q}^\dagger}{\mathbf{q}^\dagger \mathbf{p}}.$$

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$$\mathbf{p} = \frac{\partial}{\partial \mathbf{q}^\dagger}\left((z_0 - z^*)\log(\mathbf{q}^\dagger\mathbf{w}) + (z^* - \zeta_0)\log(\mathbf{q}^\dagger\mathbf{v})\right);$$

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(3) $\mathbf{B}(z^*)\mathbf{w} = \mathbf{v} \implies \mathbf{B}(z) = \mathbf{I} + \frac{1}{z-z_0}\left((z_0 - z^*)\frac{\mathbf{w}\mathbf{q}^\dagger}{\mathbf{q}^\dagger\mathbf{w}} + (z^* - \zeta_0)\frac{\mathbf{v}\mathbf{q}^\dagger}{\mathbf{q}^\dagger\mathbf{v}}\right)$,

$$\mathbf{p} = \frac{\partial}{\partial \mathbf{q}^\dagger}\left((z_0 - z^*)\log(\mathbf{q}^\dagger\mathbf{w}) + (z^* - \zeta_0)\log(\mathbf{q}^\dagger\mathbf{v})\right);$$

(4) $\mathbf{w}^\dagger\mathbf{B}(z^*) = \mathbf{v}^\dagger \implies \mathbf{B}(z) = \mathbf{I} + \frac{1}{z-z_0}\left((z_0 - z^*)\frac{\mathbf{p}\mathbf{w}^\dagger}{\mathbf{w}^\dagger\mathbf{p}} + (z^* - \zeta_0)\frac{\mathbf{p}\mathbf{v}^\dagger}{\mathbf{v}^\dagger\mathbf{p}}\right)$.

Multiplicative Representation (continued)

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Factors and Divisors

$$\begin{aligned}
\mathbf{L}(z) &= \mathbf{L}_0\mathbf{C}_1(z)\cdots\mathbf{C}_n(z), & \det\mathbf{C}_k(z) &= \frac{z-\zeta_k}{z-z_k}\\
&= \mathbf{L}^r_k(z)\mathbf{B}^r_k(z), & \det\mathbf{B}^r_k(z) &= \frac{z-\zeta_k}{z-z_k}\\
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\end{aligned}$$

Multiplicative Representation (continued)

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Definition

We call $\mathbf{C}_k(z)$ the factors of $\mathbf{L}(z)$ and $\mathbf{B}^r_k(z)$ (resp. $\mathbf{B}^l_k(z)$) right (resp. left) divisors of $\mathbf{L}(z)$ (or $\mathbf{M}(z)$).

Lemma

Let $\mathbf{L}_k = \operatorname{res}_{z_k} \mathbf{L}(z) = \mathbf{a}_k \mathbf{b}_k^\dagger$ and $\mathbf{M}_k = -\operatorname{res}_{\zeta_k} \mathbf{M}(z) = \mathbf{c}_k \mathbf{d}_k^\dagger$. Then

$$\mathbf{B}_k^r(z) = \mathbf{I} + \frac{z_k - \zeta_k}{z - z_k} \frac{\mathbf{c}_k \mathbf{b}_k^\dagger}{\mathbf{b}_k^\dagger \mathbf{c}_k}$$

$$\mathbf{B}_k^l(z) = \mathbf{I} + \frac{z_k - \zeta_k}{z - z_k} \frac{\mathbf{a}_k \mathbf{d}_k^\dagger}{\mathbf{d}_k^\dagger \mathbf{a}_k}.$$

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Proof

Let $\mathbf{L}(z) = \mathbf{L}_k^r(z)\mathbf{B}_k^r(z)$. Taking the residue at z_k gives

$$\mathbf{a}_k \mathbf{b}_k^\dagger = \mathbf{L}_k^r(z_k)\mathbf{p}_k^r(\mathbf{q}_k^r)^\dagger, \qquad \text{and so } (\mathbf{q}_k^r)^\dagger = \mathbf{b}_k^\dagger.$$

Similarly, taking the residue of $\mathbf{M}(z) = (\mathbf{B}_k^r(z))^{-1}(\mathbf{L}_k^r(z))^{-1}$ at ζ_k gives $\mathbf{p}_k^r = \mathbf{c}_k$.

Refactorization Transformations

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From now on, restrict our attention to the quadratic ($n = 2$) case:

$$\mathbf{L}(z) = \mathbf{L}_0\mathbf{C}_1(z)\mathbf{C}_2(z) = \mathbf{B}_2^l(z)\mathbf{L}_0\mathbf{B}_1^r(z) = \mathbf{B}_1^l(z)\mathbf{L}_0\mathbf{B}_2^r(z)$$

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Related isomonodromic transformation:

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Iterate it: this is our dynamics.
Question: is it possible to write down a Lagrangian generating this dymanics?

Discrete Euler-Lagrange equations

Discrete Euler-Lagrange equations

Let $\mathcal{Q}$ be the configuration space of our discrete dynamical system.

<u>Continuous Case</u>

- The Lagrangian

  $\mathcal{L} = \mathcal{L}(\mathbf{Q}, \dot{\mathbf{Q}}) \in \mathcal{F}(T\mathcal{Q})$

- Action

  $S(\gamma) = \int_\gamma \mathcal{L}(\mathbf{Q}, \dot{\mathbf{Q}})\, dt$

- Euler-Lagrange Equations (from $\delta S = 0$)

  $\frac{\partial \mathcal{L}}{\partial \mathbf{Q}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{Q}}} = 0$

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Continuous Case		Discrete Case
• The Lagrangian		
$\mathcal{L} = \mathcal{L}(\mathbf{Q}, \dot{\mathbf{Q}}) \in \mathcal{F}(T\mathcal{Q})$	$\Longrightarrow$	$\mathcal{L} = \mathcal{L}(\mathbf{Q}, \widetilde{\mathbf{Q}}) \in \mathcal{F}(\mathcal{Q} \times \mathcal{Q})$
• Action		
$S(\gamma) = \int_\gamma \mathcal{L}(\mathbf{Q}, \dot{\mathbf{Q}})\, dt$	$\Longrightarrow$	$S(\{\mathbf{Q}_k\}) = \sum_k \mathcal{L}(\mathbf{Q}_k, \mathbf{Q}_{k+1})$
• Euler-Lagrange Equations (from $\delta S = 0$)		
$\frac{\partial \mathcal{L}}{\partial \mathbf{Q}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{Q}}} = 0$	$\Longrightarrow$	$\frac{\partial \mathcal{L}}{\partial \mathbf{Y}}(\underset{\sim}{\mathbf{Q}}, \mathbf{Q}) + \frac{\partial \mathcal{L}}{\partial \mathbf{X}}(\mathbf{Q}, \widetilde{\mathbf{Q}}) = 0$

Discrete Lagrangian Integrable Systems

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Moser-Veselov Approach to Integrability of Discrete Systems

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 & & \downarrow \eta & & \downarrow \eta & & \\
\cdots & \xrightarrow{R} & \mathbf{L}(z) = \underset{\sim}{\mathbf{L}}_2(z)\underset{\sim}{\mathbf{L}}_1(z) = \mathbf{L}_1(z)\mathbf{L}_2(z) & \xrightarrow{R} & \widetilde{\mathbf{L}}(z) = \mathbf{L}_2(z)\mathbf{L}_1(z) = \widetilde{\mathbf{L}}_1(z)\widetilde{\mathbf{L}}_2(z) & \xrightarrow{R} & \cdots
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- the isospectral re-factorization map

$$R : \mathbf{L}(z) \to \tilde{\mathbf{L}}(z) = \mathbf{L}_2(z)\mathbf{L}(z)\mathbf{L}_2^{-1}(z)$$

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This is exactly our setup, the ordering of the poles determines the order of the factors.

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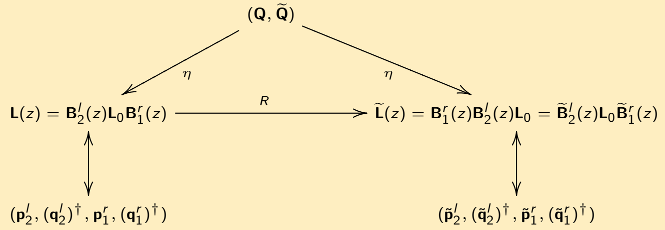

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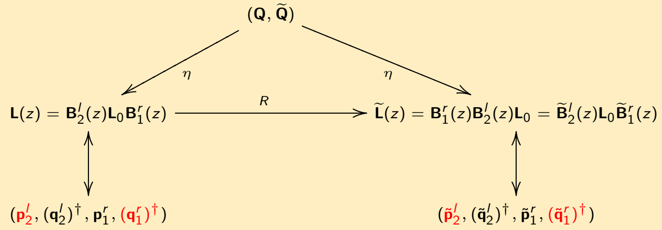

The Coordinates

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$$\eta \swarrow \qquad \searrow \eta$$

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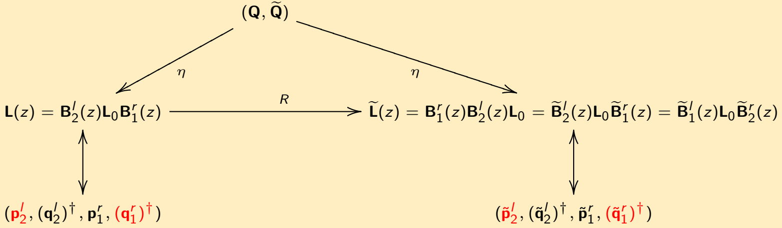

Let $\mathbf{Q} = (\mathbf{p}_2^l = \mathbf{a}_2, (\mathbf{q}_1^r)^\dagger = \mathbf{b}_1^\dagger) \in \mathcal{Q} = \mathbb{C}^m \times \mathbb{C}^m$

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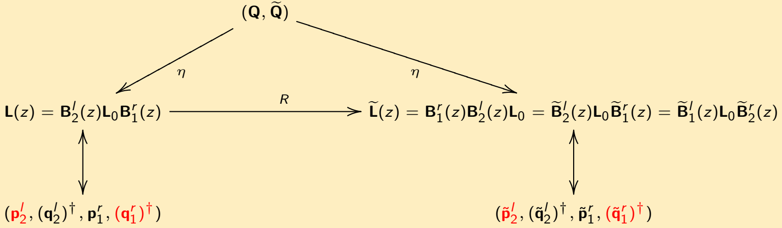

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We want:

- $\mathbf{p}_1^r = \mathbf{c}_1 = \mathbf{c}_1(\mathbf{a}_2, \mathbf{b}_1^\dagger, \tilde{\mathbf{a}}_2, \tilde{\mathbf{b}}_1^\dagger)$
- $(\mathbf{q}_2^l)^\dagger = \mathbf{d}_2^\dagger = \mathbf{d}_2^\dagger(\mathbf{a}_2, \mathbf{b}_1^\dagger, \tilde{\mathbf{a}}_2, \tilde{\mathbf{b}}_1^\dagger)$
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- $\mathbf{c}_1 = (z_1 - z_2)\dfrac{\mathbf{a}_2}{\mathbf{b}_1^\dagger \mathbf{a}_2} + (z_2 - \zeta_1)\dfrac{\tilde{\mathbf{a}}_2}{\mathbf{b}_1^\dagger \tilde{\mathbf{a}}_2}$

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- $\tilde{\mathbf{c}}_1 = (z_1 - \zeta_2)\dfrac{\mathbf{L}_0^{-1}\mathbf{a}_2}{\tilde{\mathbf{b}}_1^\dagger \mathbf{L}_0^{-1}\mathbf{a}_2} + (\zeta_2 - \zeta_1)\dfrac{\mathbf{L}_0^{-1}\tilde{\mathbf{a}}_2}{\tilde{\mathbf{b}}_1^\dagger \mathbf{L}_0^{-1}\tilde{\mathbf{a}}_2}$

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Discrete Euler-Lagrange Equations

$$-\frac{\partial \mathcal{L}}{\partial \mathbf{x}_1^\dagger}(\mathbf{Q}, \widetilde{\mathbf{Q}}) = \mathbf{c}_1 = \widetilde{\underset{\sim}{\mathbf{c}}}_1 = \frac{\partial \mathcal{L}}{\partial \mathbf{y}_1^\dagger}(\underset{\sim}{\mathbf{Q}}, \mathbf{Q})$$
$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}_2}(\mathbf{Q}, \widetilde{\mathbf{Q}}) = \mathbf{d}_2^\dagger = \widetilde{\underset{\sim}{\mathbf{d}}}_2^\dagger = -\frac{\partial \mathcal{L}}{\partial \mathbf{y}_2}(\underset{\sim}{\mathbf{Q}}, \mathbf{Q})$$

The Lagrangian

The Lagrangian

Thus, we have the following

Theorem

The equations of both the isospectral and isomonodromic dynamic can be written in the Lagrangian form with

$$\mathcal{L}(\mathbf{X},\mathbf{Y},t) = (z_2 - z_1(t))\log(\mathbf{x}_1^\dagger \mathbf{x}_2) + (z_1(t) - \zeta_2)\log(\mathbf{y}_1^\dagger \mathbf{L}_0^{-1}\mathbf{x}_2) + (\zeta_2 - \zeta_1(t))\log(\mathbf{y}_1^\dagger \mathbf{L}_0^{-1}\mathbf{y}_2) + (\zeta_1(t) - z_2)\log(\mathbf{x}_1^\dagger \mathbf{y}_2),$$

where $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2^\dagger)$ *and* $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2^\dagger)$*, in the isomonodromic case* $z_1(t) = z_1 - t$*,* $\zeta_1(t) = \zeta_1 - t$*, and in the isospectral case* $z_1(t) = z_1$*,* $\zeta_1(t) = \zeta_1$ *and* $\mathcal{L}(X,Y)$ *is time-independent.*

Sketch of the proof

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Consider $\widetilde{\mathbf{L}}(z) = \mathbf{B}_1^r(z)\mathbf{B}_2^l(z)\mathbf{L}_0 = \widetilde{\mathbf{B}}_2^l(z)\mathbf{L}_0\widetilde{\mathbf{B}}_1^r(z)$.

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- res_{z_2} : $\quad \mathbf{B}_1^r(z_2)\mathbf{a}_2\mathbf{d}_2^\dagger\mathbf{L}_0 = \tilde{\mathbf{a}}_2\tilde{\mathbf{d}}_2^\dagger\mathbf{L}_0\widetilde{\mathbf{B}}_1^r(z_2)$

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Consider $\widetilde{\mathbf{L}}(z) = \mathbf{B}_1^r(z)\mathbf{B}_2^l(z)\mathbf{L}_0 = \widetilde{\mathbf{B}}_2^l(z)\mathbf{L}_0\widetilde{\mathbf{B}}_1^r(z)$.

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Coordinates on $\mathcal{M}_{\mathcal{D}}^r$

Coordinates on $\mathcal{M}^r_{\mathcal{D}}$

Since $\underset{\sim}{\mathbf{B}^r_1(z)}\underset{\sim}{\mathbf{B}^l_2(z)}\mathbf{L}_0 = \mathbf{B}^l_1(z)\mathbf{L}_0\mathbf{B}^r_2(z)$, $\mathbf{c}_2 = \mathbf{L}_0^{-1}\underset{\sim}{\mathbf{a}}_2$, $\mathbf{d}^\dagger_1 = \underset{\sim}{\mathbf{b}}^\dagger_1$, and we have:

Theorem

The vectors $(\mathbf{c}_2, \mathbf{d}^\dagger_1; \mathbf{a}_2, \mathbf{b}^\dagger_1)$*, considered up to rescaling, are coordinates on the space* $\mathcal{M}^{\mathcal{D}}_r$*. To recover* $\mathbf{L}^{\pm 1}(z)$*, consider the function*

$$\mathcal{L}((\mathbf{x}_2, \mathbf{x}^\dagger_1), (\mathbf{y}_2, \mathbf{y}^\dagger_1)) = (z_2 - z_1)\log(\mathbf{x}^\dagger_1\mathbf{L}_0\mathbf{x}_2) + (z_1 - \zeta_2)\log(\mathbf{y}^\dagger_1\mathbf{x}_2) + (\zeta_2 - \zeta_1)\log(\mathbf{y}^\dagger_1\mathbf{L}_0^{-1}\mathbf{y}_2) + (\zeta_1 - z_2)\log(\mathbf{x}^\dagger_1\mathbf{y}_2).$$

Then

$$\mathbf{a}_1 = -\frac{\partial\mathcal{L}}{\partial\mathbf{x}^\dagger_1}((\mathbf{c}_2, \mathbf{d}^\dagger_1), (\mathbf{a}_2, \mathbf{b}^\dagger_1)); \qquad \mathbf{b}^\dagger_2 = \frac{\partial\mathcal{L}}{\partial\mathbf{x}_2}((\mathbf{c}_2, \mathbf{d}^\dagger_1), (\mathbf{a}_2, \mathbf{b}^\dagger_1));$$

$$\mathbf{c}_1 = \frac{\partial\mathcal{L}}{\partial\mathbf{y}^\dagger_1}((\mathbf{c}_2, \mathbf{d}^\dagger_1), (\mathbf{a}_2, \mathbf{b}^\dagger_1)); \qquad \mathbf{d}^\dagger_2 = -\frac{\partial\mathcal{L}}{\partial\mathbf{y}_2}((\mathbf{c}_2, \mathbf{d}^\dagger_1), (\mathbf{a}_2, \mathbf{b}^\dagger_1)).$$

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In the rank $r = 2$ case, the isomonodromic dynamics, written down in the so-called spectral coordinates, is described by the discrete Painlevé equations.

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In the rank $r = 2$ case, the isomonodromic dynamics, written down in the so-called spectral coordinates, is described by the discrete Painlevé equations.

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Main new feature of our approach is the use of rational functions, which sometimes gives computational advances, emphasis on the re-factorization, and the relationship to the Lagrangian I mentioned earlier.

Rank 2 case: general remarks

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We consider

- $\mathbf{L}(z) = \mathbf{L}_0 + \frac{\mathbf{L}_1}{z-z_1} + \frac{\mathbf{L}_2}{z-z_2}$ and $\mathbf{M}(z) = \mathbf{L}(z)^{-1} = \mathbf{M}_0 - \frac{\mathbf{M}_1}{z-\zeta_1} - \frac{\mathbf{M}_2}{z-\zeta_2}$,

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- $\mathbf{L}_0 = \operatorname{diag}\{\rho_1, \rho_2\}$, $\qquad \mathbf{M}_0 = \operatorname{diag}\{1/\rho_1, 1/\rho_2\}$,
- $\mathbf{L}_i = \alpha_i \begin{bmatrix} a_i \\ 1 \end{bmatrix} \begin{bmatrix} 1 & b_i \end{bmatrix}, \qquad \mathbf{M}_i = \beta_i \begin{bmatrix} c_i \\ 1 \end{bmatrix} \begin{bmatrix} 1 & d_i \end{bmatrix}, \qquad (i = 1, 2),$

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- $\det \mathbf{L}(z) = \rho_1 \rho_2 \frac{(z-\zeta_1)(z-\zeta_2)}{(z-z_1)(z-z_2)}$.
- Then

$$\mathbf{L}(z) = \begin{bmatrix} \rho_1 + \frac{\alpha_1 a_1}{z-z_1} + \frac{\alpha_2 a_2}{z-z_2} & \frac{\alpha_1 a_1 b_1}{z-z_1} + \frac{\alpha_2 a_2 b_2}{z-z_2} \\ \frac{\alpha_1}{z-z_1} + \frac{\alpha_2}{z-z_2} & \rho_2 + \frac{\alpha_1 b_1}{z-z_1} + \frac{\alpha_2 b_2}{z-z_2} \end{bmatrix}$$

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$$\mu(z) := \mathbf{L}(z)_{21} = \frac{\alpha_1}{z - z_1} + \frac{\alpha_2}{z - z_2} = \frac{\hat{\mu}(z - \gamma)}{(z - z_1)(z - z_2)}.$$

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$$\mathbf{L}(\gamma) = \begin{bmatrix} \rho_1\pi_1 & * \\ 0 & \rho_2\pi_2 \end{bmatrix}, \qquad \pi_1\pi_2 = \frac{(z-\zeta_1)(z-\zeta_2)}{(z-z_1)(z-z_2)}.$$

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Spectral Coordinates

- Put

$$\mu(z) := \mathbf{L}(z)_{21} = \frac{\alpha_1}{z - z_1} + \frac{\alpha_2}{z - z_2} = \frac{\hat{\mu}(z-\gamma)}{(z-z_1)(z-z_2)}.$$

- Then $\alpha_i = \operatorname{res}_{z_i} \mu(z) = \frac{\hat{\mu}(\gamma - z_i)}{(z_j - z_i)}$, $i, j = 1, 2$ and $i \neq j$.
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The Spectral Coordinates

The pair (γ, π) is called the *spectral coordinates* of $\mathbf{L}(z)$.

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Types of $\mathbf{L}(z)$ and $\mathbf{M}(z)$

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This follows from $\mathbf{M}_\infty = -\mathbf{L}_0^{-1}\mathbf{L}_\infty\mathbf{L}_0^{-1}$ and $\mathbf{M}(\gamma) = \begin{bmatrix} \frac{1}{\rho_1}\frac{1}{\pi_1} & * \\ 0 & \frac{1}{\rho_2}\frac{1}{\pi_2} \end{bmatrix}$.

$\mathbf{L}(z)$ in spectral coordinates

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Additive form of $\mathbf{L}(z)$ in spectral coordinates

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Additive form of $\mathbf{M}(z)$ in spectral coordinates

$$c_1 = \frac{\rho_2}{\mu}(k_1 - \varphi_1(\zeta_2, \zeta_2)/\pi_1) \qquad c_2 = \frac{\rho_2}{\mu}(k_1 - \varphi_1(\zeta_1, \zeta_1)/\pi_1)$$
$$d_1 = \frac{\rho_1}{\mu}(k_2 - \varphi_2(\zeta_2, \zeta_2)/\pi_2) \qquad b_2 = \frac{\rho_1}{\mu}(k_2 - \varphi_2(\zeta_1, \zeta_1)/\pi_2).$$

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Together they completely describe left and right divisors of $\mathbf{L}(z)$ and $\mathbf{M}(z)$.

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$\mathbf{L}(z)$:	z_1	z_2	ζ_1	ζ_2	ρ_1	ρ_2	k_1	k_2	μ	γ	π
$\tilde{\mathbf{L}}(z)$:	$\tilde{z}_1 = z_1 - 1$	$\tilde{z}_2 = z_2$	$\tilde{\zeta}_1 = \zeta_1 - 1$	$\tilde{\zeta}_2 = \zeta_2$	ρ_1	ρ_2	k_1	k_2	$\tilde{\mu}$	$\tilde{\gamma}$	$\tilde{\pi}$

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This explains the normalization $\pi = \pi_1 \frac{(\gamma - z_1)}{(\gamma - \zeta_2)}$:

$$\pi \tilde{\pi} = \frac{\rho_2}{\rho_1} \frac{(\tilde{\gamma} - \tilde{z}_1)(\tilde{\gamma} - \tilde{\zeta}_1)}{(\tilde{\gamma} - \tilde{z}_2)(\tilde{\gamma} - \tilde{\zeta}_2)} \qquad \text{(dPV (a))}$$

$$\tilde{\gamma} + \gamma = z_2 + \zeta_2 + \frac{k_1 + \zeta_2 - z_1}{\pi - 1} + \frac{\rho_2(k_2 - z_1 + \zeta_2 + 1)}{\rho_1 \pi - \rho_2} \qquad \text{(dPV (b))}$$

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$$\mathbf{G}_1^r = \frac{z_1 - \zeta_1}{\mathbf{b}_1^\dagger \mathbf{c}_1} \begin{bmatrix} \frac{\rho_2}{\mu}(k_1 - \varphi_1(\zeta_2, \zeta_2)/\pi_1) \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{\rho_2}{\mu}(k_2 - \varphi_2(z_2, z_2)) \end{bmatrix}$$

$$\tilde{\mathbf{G}}_1^l = \frac{z_1 - \zeta_1}{\tilde{\mathbf{d}}_1^\dagger \tilde{\mathbf{a}}_1} \begin{bmatrix} \frac{\rho_1}{\tilde{\mu}}(k_1 - \tilde{\varphi}_1(\tilde{z}_2, \tilde{z}_2)) \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{\rho_1}{\tilde{\mu}}(k_2 - \tilde{\varphi}_2(\tilde{\zeta}_2, \tilde{\zeta}_2)/\tilde{\pi}_2) \end{bmatrix},$$

the rest is a simple direct computation.