Trace identities, variational identities and Hamiltonian structures

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Outline

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Integrability problem

Given an initial value problem

$$K(u, u', \cdots) = 0, \ \ u|_{t=0} = u_0,$$

how can one determine the solution?

ODEs: Liouville-Arnold theory:

$$\text{Sufficiently many conserved quantities} \ \Rightarrow \ \text{Integrability}$$

PDEs: Integrability requires infinitely many conservation laws:

$$F_x + H_t = 0 \Rightarrow \tilde{H} = \int H\, dx \ - \ \text{conserved}$$

Spectral problem and recursion operator

$$\phi_x = U(u,\lambda)\phi \text{ or } E\phi = U(u,\lambda)\phi \quad \Leftrightarrow \quad u_t = \Phi^n K_0[u]$$

$$\Updownarrow \qquad \boxed{u_t = K_0[u] \Leftrightarrow U_t - V_x + [U,V] = 0} \qquad \Updownarrow$$

$$\text{spectral matrix } U \quad \Leftrightarrow \quad \text{recursion operator } \Phi$$

Integrable theories

- Inverse scattering transform
  Hirota's bilinear forms
  Sato's KP theory
  Wronskian and Casorati determinant techniques
  Bäcklund, Darboux and Frobenius transformations
  Singularity analysis and Painlevé property
  Symmetry and Lie group method
  etc.
- Infinitely many symmetries
  Infinitely many conservation laws
  Virasoro algebras and loop groups
  Hamiltonian structures and bi-Hamiltonian structures
  etc.

Hamiltonian structures

Continuous Hamiltonian equation:

$$u_t = K(u, u_x, \cdots) = J\frac{\delta \mathcal{H}}{\delta u}$$

where J - Hamiltonian, $\mathcal{H} = \int H[u]\, dx$.

Discrete Hamiltonian equation:

$$u_t = K(u, Eu, E^{-1}u, \cdots)] = J\frac{\delta \mathcal{H}}{\delta u}$$

where J - Hamiltonian, $\mathcal{H} = \sum_{n\in\mathbb{Z}} H[u]$.

Hamiltonian structures

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Hamiltonian properties

Relations with symmetries:

Conserved functional $\rightarrow$ adjoint symmetry $\rightarrow$ symmetry :

$$\mathcal{I} \quad \rightarrow \quad \frac{\delta \mathcal{I}}{\delta u} \quad \rightarrow \quad J\frac{\delta \mathcal{I}}{\delta u}\,.$$

Lie homomorphism : $\quad J\frac{\delta}{\delta u}\{\mathcal{I}_1,\mathcal{I}_2\} = [J\frac{\delta \mathcal{I}_1}{\delta u}, J\frac{\delta \mathcal{I}_2}{\delta u}]\,.$

Hamiltonian structures

The question:
Given a soliton equation

$$u_t = K(u) \quad \Leftrightarrow \quad U_t - V_x + [U, V] = 0,$$

how to generate its Hamiltonian structure?

$$u_t = K(u) = J\frac{\delta \mathcal{H}}{\delta u}$$

In particular, how to determine a Hamiltonian operator J?

Variational identities under bilinear forms

- **Variational identities:**

$$\frac{\delta}{\delta u}\int \langle V, U_\lambda\rangle\, dx \ [\text{or } \frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}}\langle V, U_\lambda\rangle] = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\langle V, \frac{\partial U}{\partial u}\rangle,$$

where γ - a constant, $\langle\cdot,\cdot\rangle$ - non-degenerate symmetric invariant bilinear form, and $U, V \in g$ (a Lie algebra, either semisimple or non-semisimple) satisfy

$$V_x = [U, V] \ [\text{or } (EV)(EU) = UV].$$

Trace identities under the Killing forms

- **Trace identities:**

  - G.Z. Tu, J. Phys. A 22(1989) 2375; 23(1990) 3903

  If G is a semi-simple Lie algebra, then the variational identities becomes the so-called trace identities:

$$\frac{\delta}{\delta u}\int \mathrm{tr}(VU_\lambda)\,dx\ [\mathrm{or}\ \sum_{n\in\mathbb{Z}}\mathrm{tr}(VU_\lambda)] = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\mathrm{tr}(V\frac{\partial U}{\partial u}),$$

  where γ - constant, $U, V \in g$ satisfy

$$V_x = [U,V]\ [\mathrm{or}\ (EV)(EU) = UV].$$

- **Applications:**

  KdV, AKNS, Toda lattice, Volterra lattice, etc.

Properties of bilinear forms

- **Non-degenerate property:**

  If $\langle A, B\rangle = 0$ for all A (or B), then $B = 0$ (or $A = 0$).

- **The symmetric property:**

$$\langle A, B\rangle = \langle B, A\rangle,\ \ A, B \in g.$$

- **Invariance property under the multiplication:**

$$\langle A, BC\rangle = \langle AB, C\rangle,\ \ A, B, C \in g.$$

Properties of bilinear forms

- **Invariance property under the Lie bracket:**

  If g is associative, then g forms a Lie algebra under

$$[A, B] = AB - BA.$$

  The invariance property under the Lie bracket reads

$$\langle A, [B, C] \rangle = \langle [A, B], C \rangle, \ \ A, B, C \in g.$$

- **Invariance property under isomorphisms:**

$$\langle \rho(A), \rho(B) \rangle = \langle A, B \rangle, \ \ A, B \in g,$$

  where ρ - isomorphism of g.

Two observations

- **The Killing form:**

  If g is simesimple, then all bilinear forms satisfy the above properties is equivalent to the Killing form.

- **Integrable couplings:**

  An arbitrary Lie algebra $\bar{g}$:

$$\bar{g} = g \in g_c,$$

  where g - semisimple, g_c - solvable.
  This correspond to integrable couplings.

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Formulas for the constant γ

- **The continuous case:**

  Let $V_x = [U, V]$. If $|\langle V, V\rangle| \neq 0$, then

$$\boxed{\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle V, V\rangle|.}$$

- **The discrete case:**

  Let $(EV)(EU) = UV$ and $\Gamma = VU$. If $|\langle \Gamma, \Gamma\rangle| \neq 0$, then

$$\boxed{\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle \Gamma, \Gamma\rangle|.}$$

Non-semisimple Lie algebras

As an example, take a semi-direct sum of Lie algebras $\bar{g} = g \in g_c$:

$$g = \left\{ \mathrm{diag}(A_0, A_0) \,\middle|\, A_0 = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \right\},$$

$$g_c = \left\{ \begin{bmatrix} 0 & A_1 \\ 0 & 0 \end{bmatrix} \,\middle|\, A_1 = \begin{bmatrix} a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \right\}.$$

Introduce

$$\delta : \bar{g} \to R^8,\ A \mapsto (a_1, \cdots, a_8)^T,\ A = \begin{bmatrix} A_0 & A_1 \\ 0 & A_0 \end{bmatrix} \in \bar{g}.$$

This mapping δ induces a Lie bracket on $\mathbb{R}^8$:

$$[a, b]^T = a^T R(b).$$

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Transforming basic properties of bilinear forms

An arbitrary bilinear form is given by

$$\langle a, b \rangle = a^T F b, \;\; a, b \in \mathbb{R}^8,$$

where F - constant matrix.

The symmetric property $\langle a, b\rangle = \langle b, a\rangle \Leftrightarrow F^T = F.$

The invariance property $\langle a, [b, c]\rangle = \langle [a, b], c\rangle \Leftrightarrow$

$$F(R(b))^T = -R(b)F, \;\; b \in \mathbb{R}^8.$$

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The matrix F

Solving the resulting system yields

$$F = \begin{bmatrix} \eta_1 & 0 & 0 & \eta_2 & \eta_3 & 0 & 0 & \eta_4 \\ 0 & 0 & \eta_1-\eta_2 & 0 & 0 & 0 & \eta_3-\eta_4 & 0 \\ 0 & \eta_1-\eta_2 & 0 & 0 & 0 & \eta_3-\eta_4 & 0 & 0 \\ \eta_2 & 0 & 0 & \eta_1 & \eta_4 & 0 & 0 & \eta_3 \\ \eta_3 & 0 & 0 & \eta_4 & \eta_5 & 0 & 0 & \eta_5 \\ 0 & 0 & \eta_3-\eta_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_3-\eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_4 & 0 & 0 & \eta_3 & \eta_5 & 0 & 0 & \eta_5 \end{bmatrix},$$

where η_i - arbitrary constants.

Matrix Lie algebras

Let $\bar{g} = g \Subset g_c$ be a Lie algebra of

$$A = \mathrm{diag}(A_0, A_1, \cdots, A_N) = \begin{bmatrix} A_0 & A_1 & \cdots & \cdots & A_N \\ & A_0 & A_1 & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & A_0 & A_1 \\ 0 & & & & A_0 \end{bmatrix},$$

where

$$g = \mathrm{diag}(A_0, 0, \cdots, 0), \;\; g_c = \mathrm{diag}(0, A_1, \cdots, A_N),$$

and A_i - square matrices of the same order.

Matrix Lie algebras

For

$$A = (A_0, A_1, \cdots, A_N),\ B = (B_0, B_1, \cdots, B_N) \in \bar{g},$$

the matrix product AB:

$$AB = (C_0, C_1, \cdots, C_N),\ C_k = \sum_{i+j=k} A_i B_j,\ 0 \le k \le N,$$

and the matrix commutator:

$$[A, B] = AB - BA = (\cdots, \sum_{i+j=k} [A_i, B_j], \cdots).$$

Component-trace identities and dark equations

For given $U = U(u,\lambda) = (U_0, U_1, \cdots, U_N) \in \bar{g}$, we have

$$\frac{\delta}{\delta u}\int \mathrm{tr}\,\Big(\sum_{i+j=N} V_i \frac{\partial U_j}{\partial \lambda}\Big)\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}\,\Big(\sum_{i+j=N} V_i \frac{\partial U_j}{\partial u}\Big),$$

where $V = V(u,\lambda) = (V_0, V_1, \cdots, V_N) \in \bar{g}$ solves $V_x = [U, V]$.

The case $N = 1 \Rightarrow$ Hamiltonian structures for "dark equations":

$$u_t = K(u),\ \ \psi_t = A(u,\partial_x)\psi,$$

where $A(u, \partial_x)$ - a linear differential operator.

The continuous case

Symmetry equation:

$$\rho_t = K'(u)[\rho].$$

The first-order perturbation equation:

$$u_t = K(u),\ \rho_t = K'(u)[\rho].$$

The component-trace identity with $N = 1 \Rightarrow$ a bi-trace identity:

$$\frac{\delta}{\delta u}\int\left[\mathrm{tr}\left(V_0\frac{\partial U_1}{\partial\lambda}\right)+\mathrm{tr}\left(V_1\frac{\partial U_0}{\partial\lambda}\right)\right]dx$$
$$=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left[\mathrm{tr}\left(V_0\frac{\partial U_1}{\partial u}\right)+\mathrm{tr}\left(V_1\frac{\partial U_0}{\partial u}\right)\right],$$

The discrete case

Similar results hold for the discrete case:

the component-trace identity with $N = 1$
$\Rightarrow$ a bi-trace identity:

$$\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}}\Big[\mathrm{tr}\,\Big(V_0\frac{\partial U_1}{\partial\lambda}\Big)+\mathrm{tr}\,\Big(V_1\frac{\partial U_0}{\partial\lambda}\Big)\Big]$$
$$=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\Big[\mathrm{tr}\,\Big(V_0\frac{\partial U_1}{\partial u}\Big)+\mathrm{tr}\,\Big(V_1\frac{\partial U_0}{\partial u}\Big)\Big],$$

Hamiltonian structure

The first-order perturbation equation:

$$u_t = K(u),\ \ \rho_t = K_1 = K'(u)[\rho]$$

has a Hamiltonian structure:

$$\bar{u}_t = \bar{J}\frac{\delta \bar{\mathcal{H}}}{\delta \bar{u}},\ \ \bar{J} = \left[\begin{array}{cc} 0 & J \\ J & J_1 \end{array}\right],\ \ J_1 = J'(u)[\rho],$$

$$\text{with } \bar{\mathcal{H}} = \int \mathrm{tr}(V\frac{\partial U_1}{\partial \lambda} + V_1\frac{\partial U}{\partial \lambda})\, dx\ [\text{or } \sum_{n\in \mathbb{Z}} \mathrm{tr}(V\frac{\partial U_1}{\partial \lambda} + V_1\frac{\partial U}{\partial \lambda})],$$

where $U_1 = U'(u)[\rho]$ and $V_1 = V'(u)[\rho]$.

Variational identities on Lie superalgebras

- **Variational identities:**

  Let g be a Lie superalgebra over a supercommutative ring. Then variational identities on g holds:

$$\frac{\delta}{\delta u}\int \mathrm{str}(\mathrm{ad}_V \mathrm{ad}_{\partial U/\partial\lambda})\,dx \;(\text{or } \frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}} \mathrm{str}(\mathrm{ad}_V \mathrm{ad}_{\partial U/\partial\lambda}))$$
$$= \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}(\mathrm{ad}_V \mathrm{ad}_{\partial U/\partial u}),$$

  where $U, V \in g$, $V_x = [U, V]$ (or $(EV)(EU) = UV$), $\mathrm{ad}_a b = [a, b]$, and str is the supertrace.

Super-Hamiltonian structures

- **The super-soliton hierarchy:**

$$U = U(p,q) + \alpha E_3 + \beta E_4 = \begin{bmatrix} \begin{matrix} & U(p,q) & \end{matrix} & \begin{matrix} \alpha \\ \beta \end{matrix} \\ \begin{matrix} \beta & -\alpha \end{matrix} & 0 \end{bmatrix},$$

  where E_3, E_4 - odd generators of the super $\mathrm{sl}(2)$, p, q - commuting variables and α, β - anticommuting variables.

- **Super-Hamiltonian structures:**
  Applications of super-variational identities to super-integrable systems

The super-AKNS hierarchy

- **The super-AKNS spectral problem:**

  The super AKNS spectral problem associated with $\tilde{B}(0,1)$:

$$\phi_x = U\phi = U(u,\lambda)\phi,\ \ U = \begin{bmatrix} \lambda & p & \alpha \\ q & -\lambda & \beta \\ \beta & -\alpha & 0 \end{bmatrix},\ \ u = \begin{bmatrix} p \\ q \\ \alpha \\ \beta \end{bmatrix},$$

  where p, q - commuting fields, α, β - anticommuting fields, and λ - the spectral parameter.

The super-AKNS hierarchy

- **The solution V to $V_x = [U, V]$:**

  Take a solution V as follows:

$$V = \begin{bmatrix} A & B & \rho \\ C & -A & \sigma \\ \sigma & -\rho & 0 \end{bmatrix} = \sum_{i \geq 0} V_i \lambda^{-i} = \sum_{i \geq 0} \begin{bmatrix} A_i & B_i & \rho_i \\ C_i & -A_i & \sigma_i \\ \sigma_i & -\rho_i & 0 \end{bmatrix} \lambda^{-i},$$

  where A_i, B_i, C_i are commuting fields, and ρ_i, σ_i are anticommuting fields.

- **The super-AKNS hierarchy:**

$$u_{t_m} = K_m = (-B_{m+1}, 2C_{m+1}, -\rho_{m+1}, \sigma_{m+1})^T, \ \ m \geq 0.$$

Application of the super-variational identity

- **Super-Hamiltonian structures:**
  The super-variational identity where $\gamma = 0$ leads to
$$\frac{\delta}{\delta u}\int \frac{2A_{m+1}}{m}\,dx = (-C_m, -B_m, 2\sigma_m, -2\rho_m)^T,\ \ m \geq 1.$$
  So, the super-Hamiltonian structures read
$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_m}{\delta u},\ \ m \geq 0,$$
  where J and $\mathcal{H}_m$ are
$$J = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix},\ \mathcal{H}_m = \int \frac{2A_{m+2}}{m+1}\,dx,\ \ m \geq 0.$$

Super-symmetric integrable systems:

$D = 1$ and $N = 1$ case:

How to solve

$$D_x V = [U, V], \;\; D_x = \partial_\theta + \theta \partial_x,$$

to realize

$$U_t - D_x V + [U, V] = 0?$$

Coupled equations:

The coupled perturbation system:

$$\begin{cases} u_t = K(u), \\ v_t = K'(u)[v], \\ w_t = K'(u)[w]. \end{cases}$$

Does this possess any Hamiltonian structure?

Super-integrable couplings:

Semi-direct sums of Lie superalgebras:

For example, $\bar{g} = g \Subset g_c$ with Lie product:

$$\bar{W} = W + W_c = \left[\bar{U}, \bar{V}\right] = [U + U_c, V + V_c],$$

$$W = [U, V], \ \ W_c = [U, V_c] + [U_c, V].$$

Super-integrable couplings:

Bilinear forms on semi-direct sums:

Anti-commuting variables in $\bar{g}$ bring difficulties.

Applications to super-integrable couplings:

How to determine useful super-variational identities on $\bar{g}$?

Applications to dark equations.

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Open question on linear DEs

- W.X. Ma and B. Shekhtman, *Linear Multilinear Algebra*, to appear (2009)

Consider a Cauchy problem

$$\dot{x}(t) = A(t)x(t), \ \ x(0) = x_0 \in \mathbb{R}^n.$$

$[A(t), B(t)] = 0 \ \Rightarrow \ x(t) = e^{B(t)}x_0$, where $B(t) = \int_0^t A(s)\,ds$.

The question:

Is $[A(t), B(t)] = 0$ necessary to guarantee $x(t) = e^{B(t)}x_0$?

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