Quasideterminant solutions to the Manin-Radul super KdV equation

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Outline

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- Quasideterminants and superdeterminants
- Darboux transformations in terms of a deformed derivation
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  - Darboux transformations
- The Manin-Radul super KdV equation
  - Quasideterminant solutions by Darboux transformations
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  - From quasideterminants to superdeterminants
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Motivation

- Recent interest in noncommutative version of integrable systems (Paniak, Hamanaka & Toda, Wang & Wadati, Nimmo & Gilson etc.)
- Different reasons for noncommutativity - matrix, quaternion version etc. or due to quantization (Moyal product).
- Supersymmetric equations are a particular type of noncommutativity and often have superdeterminant solutions.
- In commutative case, Darboux transformations give determinant solutions to soliton equations.
- Quasideterminants are the natural replacement when entries in a matrix do not commute.
- For matrix with supersymmetric entries, quasideterminants are related to superdeterminants.

Quasideterminants - Definition

Developed since early 1990s by Gelfand and Retakh; recent review article Gelfand et al (2005) *Advances in Mathematics*, **193**, 56-141.

Definition

An $n \times n$ matrix $A = (a_{i,j})$ over a ring (non-commutative, in general) has n^2 *quasideterminants* written as $|A|_{i,j}$. Defined recursively by

$$|A|_{i,j} = a_{i,j} - r_i^j (A^{i,j})^{-1} c_j^i, \quad A^{-1} = (|A|_{j,i}^{-1})_{i,j=1,\dots,n}.$$

Notation: $A = \begin{bmatrix} A^{i,j} & c_j^i & \\ r_i^j & a_{i,j} & \\ & & \end{bmatrix}$

Quasideterminants - Noncommutative Jacobi Identity

Noncommutative Jacobi identity

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}$$

C.f. Jacobi identity

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & i \end{vmatrix} = \begin{vmatrix} A & C \\ E & i \end{vmatrix} \begin{vmatrix} A & B \\ D & f \end{vmatrix} - \begin{vmatrix} A & B \\ E & h \end{vmatrix} \begin{vmatrix} A & C \\ D & g \end{vmatrix}$$

Quasideterminants - Invariance

The following formula can be used to understand the effect on a quasideterminant of certain elementary row operations involving addition and multiplication on the left

$$\left|\begin{pmatrix} E & 0 \\ F & g \end{pmatrix}\begin{pmatrix} A & B \\ C & d \end{pmatrix}\right|_{n,n} = \begin{vmatrix} EA & EB \\ FA+gC & \boxed{FB+gd} \end{vmatrix} = g\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}.$$

There is analogous invariance under column operations involving addition and multiplication on the right.

Remark. This property is very important for re-ordering a quasideterminant to get an even super matrix and determining the parity of a quasideterminant.

Quasideterminants - Applications to linear systems

Solutions of systems of linear systems over an arbitrary ring can be expressed in terms of quasideterminants.

Theorem 1. Let $A = (a_{ij})$ be an $n \times n$ matrix over a ring $\mathcal{R}$. Assume that all the quasideterminants $|A|_{ij}$ are defined and invertible. Then the system of equations

$$x_1 a_{1i} + x_2 a_{2i} + \cdots + x_n a_{ni} = b_i, \;\; 1 \leq i \leq n \tag{1}$$

has the unique solution

$$x_i = \sum_{j=1}^{n} b_j |A|_{ij}^{-1}, \;\; i = 1, \ldots, n. \tag{2}$$

Let $A_l(b)$ be the $n \times n$ matrix obtained by replacing the l-th row of the matrix A with the row $(b_1, \ldots, b_n)$. Then we have the following *Cramer's rule*.

Theorem 2. In notation of Theorem 1, if the quasideterminants $|A|_{ij}$ and $|A_i(b)|_{ij}$ are well defined, then

$$x_i |A|_{ij} = |A_i(b)|_{ij}.$$

Superdeterminants - Definition

In the context of superalgebra, a (block) supermatrix $\mathcal{M} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$ is said to be even if X and T are *even* square matrices and Y, Z are (not necessarily square) odd matrices. If X is $m \times m$ and T is $n \times n$ then $\mathcal{M}$ is called an (m, n)-supermatrix.

The *superdeterminant*, or *Berezinian*, of $\mathcal{M}$ is defined to be

$$\mathrm{Ber}(\mathcal{M}) = \mathrm{sdet}(\mathcal{M}) = \frac{\det(X - YT^{-1}Z)}{\det(T)} = \frac{\det(X)}{\det(T - ZX^{-1}Y)}.$$

- Berezin F.A., Introduction to superanalysis (D. Reidel Publishing Company, Dordrecht, 1987).
- DeWitt B., Supermanifolds (Cambridge University Press, 1984).

Superdeterminants vs. Quasideterminants

Lemma 1. Let $\mathcal{M}$ be an (m,n)-supermatrix. Then

$$|\mathcal{M}|_{i,j} = \begin{cases} (-1)^{i+j} \dfrac{\mathsf{Ber}(\mathcal{M})}{\mathsf{Ber}(\mathcal{M}^{i,j})}, & 1 \le i,j \le m, \\ (-1)^{i+j} \dfrac{\mathsf{Ber}(\mathcal{M}^{i,j})}{\mathsf{Ber}(\mathcal{M})}, & m+1 \le i,j \le m+n, \end{cases} \tag{3}$$

where $\mathcal{M}^{i,j}$ is the submatrix obtained by deleting row i and column j in $\mathcal{M}$. C.f. In commutative case,

$$|A|_{i,j} = (-1)^{i+j} \frac{\det(A)}{\det(A^{i,j})}.$$

- Bergvelt M.J. and Rabin J.M., Super curves, their Jacobians and super KP equations. arXiv: alg-geom/9601012v1.

A deformed derivation - Definition

Definition

Let $\mathcal{A}$ be an associative, unital algebra over ring K. An operator $D\colon \mathcal{A} \to \mathcal{A}$ satisfying $D(K) = 0$ and $D(ab) = D(a)b + h(a)D(b)$ is called a *deformed derivation*, where $h\colon \mathcal{A} \to \mathcal{A}$ is a *homomorphism*, i.e. for all $\alpha \in K$, $a, b \in \mathcal{A}$, $h(\alpha a) = \alpha h(a)$, $h(a+b) = h(a) + h(b)$ and $h(ab) = h(a)h(b)$.

Examples: We assume that elements in $\mathcal{A}$ depend on a variable x.

1. **Normal derivative** $D = \partial/\partial x$ satisfying $D(ab) = D(a)b + aD(b)$ with $h = id_{\mathcal{A}}$.
2. **Forward difference** $D(a) = \alpha^{-1}\Delta(a) = (a(x+1) - a(x))/\alpha$ satisfying $D(a(x)b(x)) = D(a(x))b(x) + a(x+1)D(b(x))$ with $h = T$ (the shift map).
3. **q-derivative** $D(a) = D_q(a) = \dfrac{a(qx) - a(x)}{(q-1)x}$ with $h(a(x)) = S_q(a) = a(qx)$.
4. **Superderivative** $D = \partial_\theta + \theta\partial_x$ satisfying $D(ab) = D(a)b + \hat{a}D(b)$ where $h = \hat{}$ is the grade involution: let a be even, b odd, then $\widehat{a+b} = a - b$.

A deformed derivation - Continued

Lemma 2.

1. Let A, B be matrices over $\mathcal{A}$. Whenever AB is defined, $h(AB) = h(A)h(B)$ and $D(AB) = D(A)B + h(A)D(B)$,
2. Let A be an invertible matrix over $\mathcal{A}$. Then $h(A)^{-1} = h(A^{-1})$ and $D(A^{-1}) = -h(A)^{-1}D(A)A^{-1}$,
3. Let A, B, C be matrices over $\mathcal{A}$ such that $AB^{-1}C$ is well-defined. Then $D(AB^{-1}C) = D(A)B^{-1}C + h(A)h(B)^{-1}(D(C) - D(B)B^{-1}C)$.

A deformed derivation - Darboux transformations

Define $G_a\colon \mathcal{A} \to \mathcal{A}$ by $G_a(b) = h(a)D(a^{-1}b) = D(b) - D(a)a^{-1}b$ for any $a \in \mathcal{A}$, then we have Darboux transformations

Theorem 3. Given $\phi, \theta_0, \theta_1, \theta_2, \dots \in \mathcal{A}$ where θ_i are invertible, the sequence of Darboux transformations of $\phi[k] \in \mathcal{A}$ is defined recursively by $\phi[k+1] = G_{\theta[k]}(\phi[k])$, where $\phi[0] = \phi$, $\theta[0] = \theta_0$ and $\theta[k] = \phi[k]\big|_{\phi \to \theta_k}$.

For example, the Darboux transformation for $k = 0$ is given by

$$\phi[1] = D(\phi) - D(\theta_0)\theta_0^{-1}\phi.$$

Remark. The formulae for the iteration of Darboux transformations are identical with those in the standard case of a regular derivation.

A deformed derivation - Darboux transformations

Theorem 4. For integers $n \geq 0$,

$$\phi[n] = \begin{vmatrix} \theta_0 & \cdots & \theta_{n-1} & \phi \\ D(\theta_0) & \cdots & D(\theta_{n-1}) & D(\phi) \\ \vdots & & \vdots & \vdots \\ D^{n-1}(\theta_0) & \cdots & D^{n-1}(\theta_{n-1}) & D^{n-1}(\phi) \\ D^n(\theta_0) & \cdots & D^n(\theta_{n-1}) & \boxed{D^n(\phi)} \end{vmatrix}.$$

Remark. The form of this iteration formula for Darboux transformations is the same as the standard one in which $D = \partial$.

The Manin-Radul super KdV equation

As a particular example, we consider the **Manin-Radul super KdV equation**

$$\partial_t \alpha = \frac{1}{4}\partial(\partial^2\alpha + 3\alpha D\alpha + 6\alpha u),$$
$$\partial_t u = \frac{1}{4}\partial(\partial^2 u + 3u^2 + 3\alpha D u),$$

Lax pair

$$\partial^2\phi + \alpha D\phi + u\phi - \lambda\phi = 0,$$
$$\partial_t\phi - \frac{1}{2}\alpha\partial D\phi - \lambda\partial\phi - \frac{1}{2}u\partial\phi + \frac{1}{4}(\partial a)D\phi + \frac{1}{4}(\partial u)\phi = 0.$$

- Y.I. Manin and A. O. Radul, *Comm. Math. Phys.* **98**(1985) 65-77.

Quasideterminant solutions by Darboux transformations

Let θ_i, $i = 0, \ldots, n-1$ be a particular set of eigenfunctions of the Lax pair. To make sense, we choose θ_i to be even if its index is even, otherwise, θ_i is odd. The Darboux transformation is then defined recursively by

$$\phi[k+1] = D(\phi[k]) - D(\theta[k])\theta[k]^{-1}\phi[k],$$
$$\alpha[k+1] = -\alpha[k] + 2\partial(D(\theta[k])\theta[k]^{-1}),$$
$$u[k+1] = u[k] + D(\alpha[k]) - 2D(\theta[k])\theta[k]^{-1}(\alpha[k] - \partial(D(\theta[k])\theta[k]^{-1})),$$

where $\phi[0] = \phi$, $\theta[0] = \theta_0$, $\alpha[0] = \alpha$, $u[0] = u$ and

$$\theta[k] = \phi[k]|_{\phi \to \theta_k}.$$

- Q.P. Liu and M. Mañas, *Physics Letters B* **396**(1997) 133-140.

Quasideterminant solutions by Darboux transformations

We introduce the quasideterminants

$$Q_n(i,j) = \begin{vmatrix} \theta_0 & \cdots & \theta_{n-1} & 0 \\ D\theta_0 & \cdots & D\theta_{n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ D^{n-j-1}\theta_0 & \cdots & D^{n-j-1}\theta_{n-1} & 1 \\ D^{n-j}\theta_0 & \cdots & D^{n-j}\theta_{n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ D^{n-1}\theta_0 & \cdots & D^{n-1}\theta_{n-1} & 0 \\ D^{n+i}\theta_0 & \cdots & D^{n+i}\theta_{n-1} & \boxed{0} \end{vmatrix}.$$

Observation 1. $h(Q_n(i,j)) = (-1)^{i+j+1}Q_n(i,j)$, that is, $Q_n(i,j)$ has the parity $(-1)^{i+j+1}$.

Observation 2. $\partial Q_n(i,j) = D^2 Q_n(i,j)$ and
$DQ_n(i,j) = Q_n(i+1,j) + (-1)^{i+j+1}Q_n(i,j+1) + (-1)^{i+1}Q_n(i,0)Q_n(0,j)$.

Quasideterminant solutions by Darboux transformations

Lemma 3.
$$D(\theta_0)\theta_0^{-1} = -Q_1(0,0),$$
$$D(\theta[k])\theta[k]^{-1} = -Q_k(0,0) - Q_{k+1}(0,0), \quad k \geq 1.$$

Theorem 4. After n repeated Darboux transformations, the Manin-Radul super KdV equation has new solutions $\alpha[n]$ and $u[n]$ expressed in terms of quasideterminants

$$\alpha[n] = (-1)^n \alpha - 2\partial Q_n(0,0),$$
$$u[n] = u - 2\partial Q_n(0,1) - 2Q_n(0,0)((-1)^n \alpha - \partial Q_n(0,0)) + \frac{1-(-1)^n}{2} D\alpha.$$

Proof. By induction.

Direct Approach

Under the assumptions $\alpha = 0$ and $u = 0$, we can prove $\alpha[n] = -2\partial Q_n(0,0)$ and $u[n] = -2\partial Q_n(0,1) + 2Q_n(0,0)\partial Q_n(0,0)$ with $\partial_t \theta_i = \partial^3 \theta_i$ $(i = 0, \ldots, n-1)$ satisfy the super KdV equation by a direct approach. To achieve this, we introduce an auxiliary variable y such that $\partial_y \theta_i = \partial^2 \theta_i$. By doing this, we can find hidden identities by letting $\partial_y \Omega_n(i,j) = 0$.

Observation 3. Through detailed calculations, we have

$$
\begin{aligned}
\partial_y Q_n(i,j) = {} & Q_n(i+4,j) - Q_n(i,j+4) + Q_n(i,0)Q_n(3,j) \\
& + Q_n(i,1)Q_n(2,j) + Q_n(i,2)Q_n(1,j) + Q_n(i,3)Q_n(0,j), \\
\partial_t Q_n(i,j) = {} & Q_n(i+6,j) - Q_n(i,j+6) + Q_n(i,0)Q_n(5,j) + Q_n(i,1)Q_n(4,j) \\
& + Q_n(i,2)Q_n(3,j) + Q_n(i,3)Q_n(2,j) + Q_n(i,4)Q_n(1,j) + Q_n(i,5)Q_n(0,j).
\end{aligned}
$$

By substitution and letting $\partial_y Q_n(i,j) = 0$ for all $i + j \leq 5$, $i \geq 0$, $j \geq 0$, all terms in the super KdV equation cancel identically.

- C.R. Gilson and J.J.C. Nimmo, On a direct approach to quasideterminant solutions of a noncommutative KP equation, *J. Phys. A: Math. Theor.* **40**(2007) 3839-3850.

From quasideterminants to superdeterminants

In Liu and Mañas' paper we mentioned before, the solutions to the super KdV system were given as

$$\alpha[n] = (-1)^n\alpha - 2\partial a_{n,n-1},$$
$$u[n] = u - 2\partial a_{n,n-2} - a_{n,n-1}((-1)^n\alpha + \alpha[n]) + \frac{1-(-1)^n}{2}D\alpha,$$

where $a_{n,n-1}, a_{n,n-2}, \dots, a_{n,0}$ satisfy the linear system

$$T_n\theta_j = (D^n + a_{n,n-1}D^{n-1} + \cdots + a_{n,0})\theta_j = 0, \;\; i = 0, \dots, n-1.$$

By solving the above linear system using **Theorem 2**, we managed to obtain a unified formula for all $a_{n,n-i}$, that is,

$$\boxed{a_{n,n-i} = Q_n(0, i-1), \quad i = 1, \dots, n},$$

which coincide with the solutions shown before when $i = 1$.

From quasideterminants to superdeterminants

To identify quasideterminant solutions with superdeterminant solutions given by Liu and Mañas, we will split (**??**) into two cases.

Case I. For $n = 2k$, denote $\mathbf{b} = (D^{2k}\theta_0, \cdots, D^{2k}\theta_{2k-2}, D^{2k}\theta_1, \cdots, D^{2k}\theta_{2k-1})$,

$$\mathcal{W} = \begin{pmatrix} \theta_0 & \cdots & \theta_{2k-2} & \theta_1 & \cdots & \theta_{2k-1} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ D^{2k-2}\theta_0 & \cdots & D^{2k-2}\theta_{2k-2} & D^{2k-2}\theta_1 & \cdots & D^{2k-2}\theta_{2k-1} \\ D\theta_0 & \cdots & D\theta_{2k-2} & D\theta_1 & \cdots & D\theta_{2k-1} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ D^{2k-1}\theta_0 & \cdots & D^{2k-1}\theta_{2k-2} & D^{2k-1}\theta_1 & \cdots & D^{2k-1}\theta_{2k-1} \end{pmatrix},$$

and $\hat{\mathcal{W}}$ is obtained from $\mathcal{W}$ by replacing the k-th row with $\mathbf{b}$, then we have

$$a_{2k,2k-1} = Q_{2k}(0,0) = D\ln(\mathrm{Ber}(\mathcal{W})), \; a_{2k,2k-2} = Q_{2k}(0,1) = -\frac{\mathrm{Ber}(\hat{\mathcal{W}})}{\mathrm{Ber}(\mathcal{W})}.$$

From quasideterminants to superdeterminants

Case II. For $n = 2k+1$, denote

$$\mathbf{c} = (D^{2k+1}\theta_0, \cdots, D^{2k+1}\theta_{2k}, D^{2k+1}\theta_1, \cdots, D^{2k+1}\theta_{2k-1}),$$

$$\mathcal{W} = \begin{pmatrix} \theta_0 & \cdots & \theta_{2k} & \theta_1 & \cdots & \theta_{2k-1} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ D^{2k}\theta_0 & \cdots & D^{2k}\theta_{2k} & D^{2k}\theta_1 & \cdots & D^{2k}\theta_{2k-1} \\ D\theta_0 & \cdots & D\theta_{2k} & D\theta_1 & \cdots & D\theta_{2k-1} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ D^{2k-1}\theta_0 & \cdots & D^{2k-1}\theta_{2k} & D^{2k-1}\theta_1 & \cdots & D^{2k-1}\theta_{2k-1} \end{pmatrix},$$

and $\hat{\mathcal{W}}$ is obtained from $\mathcal{W}$ by replacing the $(2k+1)$-th row with **c**.

From quasideterminants to superdeterminants

Liu and Mañas gave an expression for $a_{2k+1,2k}$ as the ratio of determinants rather than superdeterminants. Here we obtain an expression as the logarithmic superderivative of a superdeterminant.

$$a_{2k+1,2k-1} = Q_{2k+1}(0,1) = -\frac{\mathrm{Ber}(\mathcal{W})}{\mathrm{Ber}(\hat{\mathcal{W}})}$$

$$a_{2k+1,2k} = Q_{2k+1}(0,0) = -D\ln(\mathrm{Ber}(\mathcal{W}))$$

In contrast with the expression

$$a_{2k+1,2k} = -\frac{\det(\hat{W}^{(0)} - \hat{W}^{(1)}(D\widetilde{W}^{(1)})^{-1}(D\widetilde{W}^{(0)}))}{\det(W^{(0)} - W^{(1)}(D\widetilde{W}^{(1)})^{-1}(D\widetilde{W}^{(0)}))}$$

found by Liu and Mañas.

Conclusions

1. A deformed derivation is defined and its Darboux transformation in terms of quasideterminants is constructed.
2. As an application, quasideterminant solutions for the Manin-Radul super KdV system are obtained and proved both by induction and by direct approach.
3. By using quasideterminants, we obtain a unified expression for the solutions constructed by Darboux transformations. This also allows us to obtain solutions in terms of superdeterminants for all cases.