Recent results on multiscale technique and integrability of partial difference equations

Decio Levi
(joint work with R. Hernandez Heredero, M. Petrera and C. Scimiterna)

Electronic Engineering Department, Roma Tre University and INFN, Sezione Roma Tre

International Workshop on Nonlinear and Modern Mathematical Physics
Beijing, July 15-21, 2009

Outline

1. Introduction
   - Multiscale analysis and Integrability for PDEs
   - Multiscale on the lattice
     - from shifts to derivatives
     - from derivative to shifts
2. Integrability of discrete Nonlinear Schrödinger Equations
3. Other examples
4. Classification of lattice equations on the square
5. Conclusions

Introduction

- Multiscale analysis: perturbation technique for constructing *uniformly* valid approximation to solutions of perturbation problems;
- *Nonuniformity* arises from *secularity*.
- Multiscale perturbation methods have been introduces by Poincaré to deal with secularity problems in the perturbative solution of differential equations.
- In the reductive perturbation method introduced by Taniuti et. al., the space and time coordinates are stretched in terms of a small expansion parameter and we look for the far field behaviour of the system.
- Multi-scale expansions can be applied to both *integrable* and *non-integrable systems*.

Multiscale analysis and integrability

- Multiscale analysis: perturbation technique for testing *integrability* of a given nonlinear system [Calogero];
- *Integrability* is *preserved* in the reduction process [Zakharov, Kuznetsov *PDE*].
- Partial differential equation example: *KdV* equation for $u(x,t) \in \mathcal{R}$

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = u \frac{\partial u}{\partial x}.$$

- Solution of the form

$$u(x,t;\varepsilon) = \sum_{n=1}^{+\infty} \sum_{\alpha=-n}^{n} \varepsilon^n u_n^{(\alpha)}(\xi, t_1, t_2, \ldots) \, e^{i\alpha(\kappa x - \omega t)}.$$

$u_n^{(-\alpha)} = \bar{u}_n^{(\alpha)}$. $\xi \doteq \varepsilon x$, $t_j \doteq \varepsilon^j t$, $j \geq 1$ are the *slow variables;*

Multiscale analysis and integrability

- Space and time partial derivatives becomes:

$$\partial_x \to \partial_x + \varepsilon \partial_\xi ,$$

$$\partial_t \to \partial_t + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + \dots ,$$

  and all the variables are considered to be *independent;*

- Order ε:
  $\alpha = 1$: *dispersion relation* $\omega = -\kappa^3$;
- Order ε^2:
  $\alpha = 0$:

$$\partial_{t_1} u_1^{(0)} = 0.$$

  $\alpha = 1$:

$$\left[\partial_{t_1} + \mathrm{i}\kappa \left(3\mathrm{i}\kappa \partial_\xi - u_1^{(0)} \right) \right] u_1^{(1)} = 0.$$

  Solution:

$$u_1^{(1)} = g_1^{(1)} (\rho, t_2, t_3)\, e^{-\frac{\mathrm{i}}{3\kappa} \int_{\xi_0}^{\xi} u_1^{(0)}(\xi', t_2, t_3) d\xi'}, \qquad \rho \doteq \xi + 3\kappa^2 t_1 .$$

$\alpha = 2$:

$$u_2^{(2)} = -\frac{1}{6\kappa^2}\left(u_1^{(1)}\right)^2;$$

- Order ε^3:

  $\alpha = 0$:

$$\partial_{t_1} u_2^{(0)} = \partial_\rho \left(|u_1^{(1)}|^2\right) + \frac{1}{2}\partial_\xi \left[\left(u_1^{(0)}\right)^2\right] - \partial_{t_2} u_1^{(0)}.$$

No-secularity conditions

- The right-hand side solves the homogeneous equation: secularity!

$$\partial_{t_1} u_2^{(0)} = \partial_\rho \left(|u_1^{(1)}|^2\right),$$

$$\left(\partial_{t_2} - u_1^{(0)}\partial_\xi\right) u_1^{(0)} = 0, \quad \textit{Hopf equation: wave breaking!}$$

Solutions:

$$u_2^{(0)} = \frac{|u_1^{(1)}|^2}{3\kappa^2}, \qquad u_1^{(0)} = 0.$$

Multiscale analysis and integrability

$\alpha = 1$:

$$\left(\partial_{t_1} - 3\kappa^2\partial_\xi\right) u_2^{(1)} = -\left(\partial_{t_2} + 3\mathrm{i}\kappa\partial_\rho^2 - \frac{\mathrm{i}}{6\kappa}|u_1^{(1)}|^2\right) u_1^{(1)}.$$

No-secularity condition

- The right-hand side solves the homogeneous equation: secularity!

$$\left(\partial_{t_1} - 3\kappa^2\partial_\xi\right) u_2^{(1)} = 0,$$

$$\left(\partial_{t_2} + 3\mathrm{i}\kappa\partial_\rho^2 - \frac{\mathrm{i}}{6\kappa}|u_1^{(1)}|^2\right) u_1^{(1)} = 0 : \quad \textit{NLS equation.}$$

- *KdV* equation and *NLS* equation are *both integrable!*

Multiscale analysis and integrability

- Higher orders beyond *NLS* equation [Degasperis, Manakov, Santini]: *fundamental* for an integrability test.

  Proposition [Degasperis, Procesi]: *If an equation is integrable, then under a multiscale expansion the functions* $u_m^{(1)}$, $m \geq 1$ *satisfy the equations*

$$\partial_{t_n} u_1^{(1)} = K_n \left[u_1^{(1)} \right],$$

$$M_n u_j^{(1)} = g_n(j), \qquad M_n \doteq \partial_{t_n} - K_n' \left[u_1^{(1)} \right],$$

  $\forall\, j,\; n \geq 2.$

  $K_n \left[u_1^{(1)} \right]$: *n-th flow in a hierarchy of integrable equations;*

  $K_n' \left[u_j^{(1)} \right] v$: *Frechet derivative of* $K_n[u_j^{(1)}]$ *along* v: *linearization;*

  $g_n(j)$: nonhomogeneous forcing term in a well defined polynomial vector space or *linear* combination of basic monomials.

Multiscale analysis and integrability

- *Compatibility* conditions:

$$M_k g_n(j) = M_n g_k(j), \quad \forall\, k, n \geq 2.$$

- *Integrability* conditions: set of relations among the coefficients of $g_n(j)$.
- *Definition* [Degasperis, Procesi]: *If the compatibility conditions are satisfied up to the index $j \geq 2$, our equation is asymptotically integrable of degree j (A_j integr.).*
- Known results for A_3 integrability conditions:

weakly dispersive nonlinear systems: *KdV/pot.KdV* hierarchies,

strongly dispersive nonlinear systems: *NLS* hierarchy,

their *linearizable* limits.

Multiscale on the lattice: from shifts to derivatives

Let us consider a function $u_n : \mathbb{Z} \to \mathbb{R}$ depending on a discrete index $n \in \mathbb{Z}$

- The dependence of u_n on n is realized through the *slow variable* $n_1 \doteq \varepsilon n \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$, $\varepsilon = 1/N$, $N >> 0$, $0 < \varepsilon \ll 1$, that is to say $u_n \doteq u(n_1)$;
- The variable n_1 can vary in a region of the integer axis such that $u(n_1)$ is therein analytical (Taylor series expandible);
- The radius of convergence of the Taylor series in n_1 is wide enough to include as *inner points* the points $n_1 \pm k\varepsilon$.

$T_n u_n \doteq u_{n+1} = u(n_1 + \varepsilon)$,

$T_n u(n_1) = u(n_1) + \varepsilon u^{(1)}(n_1) + \frac{\varepsilon^2}{2} u^{(2)}(n_1) + ... + \frac{\varepsilon^i}{i!} u^{(i)}(n_1) + ... = e^{\varepsilon d_{n_1}} u(n_1)$,

$$u_n \doteq u(n, n_1), \qquad T_n = \mathcal{T}_n \mathcal{T}_{n_1}^{(\varepsilon_{n_1})} = \mathcal{T}_n \sum_{j=0}^{+\infty} \varepsilon^j \mathcal{A}_n^{(j)}, \qquad \mathcal{A}_n^{(j)} \doteq \frac{N_1^j}{j!} \partial_{n_1}^j, \tag{1}$$

$$u\left(n, m, n_1, \{m_j\}_{j=1}^K, \varepsilon\right) = \sum_{\gamma=1}^{+\infty} \sum_{\alpha=-\gamma}^{\gamma} \varepsilon^\gamma u_\gamma^{(\alpha)} \left(n_1, \{m_j\}_{j=1}^K\right) E_{n,m}^\alpha, \tag{2}$$

$$E_{n,m} \doteq e^{i[\kappa n - \omega(\kappa) m]}, \qquad u_\gamma^{(-\alpha)} = \bar{u}_\gamma^{(\alpha)}$$

Multiscale on the lattice: from derivatives to shifts

Our multiscale approach produces from a given partial difference equation a partial differential equation for one of the amplitudes $u_\gamma^{(\alpha)}$. From the PDE we get a PΔE inverting the shift operator.

$$\partial_{n_1} = ln\mathcal{T}_{n_1} = ln\left(1 + h_1 \Delta_{n_1}^{(+)}\right) \doteq \sum_{i=1}^{+\infty} \frac{(-1)^{i-1} h_1^i}{i} \Delta_{n_1}^{(+)i}, \tag{3}$$

where $\Delta_{n_1}^{(+)} \doteq \frac{\mathcal{T}_{n_1}-1}{h_1}$ is *forward* difference operator in n_1.

$$\Delta_{n_1}^j u_{n_1} \doteq \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} u_{n_1+i} = \sum_{i=j}^{\infty} \frac{j!}{i!} P_{i,j}\, \Delta_n^i u_n. \tag{4}$$

$$P_{i,j} \doteq \sum_{k=j}^{i} \Omega^k \mathcal{S}_i^k\, \mathfrak{S}_k^j,$$

Ω is the ratio of the increment in the lattice of variable n with respect to that of variable n_1. The coefficients $\mathcal{S}_i^k$ and $\mathfrak{S}_k^j$ are the Stirling numbers of the first and second kind respectively.

Multiscale on the lattice: from derivatives to shifts

This is one of the possible inversion formulae for $\mathcal{T}_{n_1}$. Ex. for *symmetric* difference operator $\Delta_{n_1}^{(s)} \doteq \left(\mathcal{T}_{n_1} - \mathcal{T}_{n_1}^{-1}\right)/2h_1$ we get

$$\partial_{n_1} = sinh^{-1} h_1 \Delta_{n_1}^{(s)} \doteq \sum_{i=1}^{+\infty} \frac{P_{i-1}(0) h_1^i}{i} \Delta_{n_1}^{(s)i}, \qquad (5)$$

where $P_i(0)$ is the i-th *Legendre* polynomial evaluated in $x = 0$.

Difference equations of ∞ order. Only if u_n is a slow–varying function of order l, i.e.

$$\Delta^{l+1} u_n \approx 0$$

∂_{n_1} operator reduces to polynomials in the Δ_{n_1} of order at most l.

Integrability of discrete *NLS* equations (*dNLS*)

- The nonintegrable standard *dNLSE*

$$i\dot{u}_n + \frac{1}{2\sigma^2}\left(u_{n+1} - 2u_n + u_{n-1}\right) = \varepsilon |u_n|^2 u_n, \quad \varepsilon \doteq \pm 1, \tag{6}$$

- The integrable Ablowitz-Ladik *dNLSE*

$$i\dot{u}_n + \frac{1}{2\sigma^2}\left(u_{n+1} - 2u_n + u_{n-1}\right) = \varepsilon |u_n|^2 \left(u_{n+1} + u_{n-1}\right), \quad \varepsilon \doteq \pm 1, \tag{7}$$

- The saturable *dNLSE*

$$i\dot{u}_n + \frac{1}{2\sigma^2}\left(u_{n+1} - 2u_n + u_{n-1}\right) = \frac{|u_n|^2}{\varepsilon + |u_n|^2} u_n, \quad \varepsilon \doteq \pm 1, \tag{8}$$

- The Salerno *dNLSE*

$$i\dot{u}_n + \frac{1}{2\sigma^2}\left(u_{n+1} - 2u_n + u_{n-1}\right)\left(1 - s\varepsilon\sigma^2 |u_n|^2\right) = \varepsilon |u_n|^2 u_n, \quad \varepsilon \doteq \pm 1, \quad s \in \mathcal{R}, \tag{9}$$

interpolates between Eq. (6) when $s = 0$ and Eq. (7) when $s = 1$.

Integrability of discrete *NLS* equations (*dNLS*)

- Differential-difference equation example:

$$\mathrm{i}\dot{u}_n + \frac{1}{2\sigma^2}\left(u_{n+1} - 2u_n + u_{n-1}\right) = |u_n|^2\left(\beta_1 u_n + \beta_2 u_{n+1} + \beta_3 u_{n-1}\right) + \\ + |u_n|^4\left(\theta_1 u_n + \theta_2 u_{n+1} + \theta_3 u_{n-1}\right),$$

Ablowitz-Ladik integr. *dNLS* when $\beta_1 = \theta_1 = \theta_2 = \theta_3 = 0$ and $\beta_2 = \beta_3 = \varepsilon$;
the standard nonintegrable *dNLSE* when $\beta_2 = \beta_3 = \theta_1 = \theta_2 = \theta_3 = 0$, and $\beta_1 = \varepsilon$;
the first term of the small amplitude approximation of the saturable *dNLSE* when $\beta_1 = \varepsilon$, $\theta_1 = -1$ and $\beta_j = \theta_j = 0$, $j = 2, 3$;
the Salerno *dNLSE* when $\beta_1 = \varepsilon\left(1 - s\right)$ and $\beta_2 = \beta_3 = \varepsilon s/2$.

Integrability of discrete *NLS* equations (*dNLS*)

- Solution of the form:

$$u_n\left(t;\varepsilon\right)=\sum_{j=1}^{+\infty}\sum_{\alpha=-j}^{j}\varepsilon^j f_j^{(\alpha)}\left(n_1,t_1,t_2,\ldots\right)e^{\mathrm{i}\alpha(\kappa n-\omega t)}.$$

Expansion Parameters

1. $0\le\varepsilon\ll1$: perturbative parameter around plane wave solution of *dNLS*;
2. $n_1\doteq\varepsilon n$: slow "space" variable;
3. $t_j=\varepsilon^j t$, $j\geq1$ slow times variables;

- $f_j^{(\alpha)}(n_1, t_1, t_2, \ldots)$ $\mathcal{C}^{(\infty)}$ in n_1:

$$f_j^{(\alpha)}(n_1 \pm \varepsilon) = f_j^{(\alpha)}(n_1) \pm \varepsilon \partial_{n_1} f_j^{(\alpha)} + \frac{(\varepsilon \partial_{n_1})^2}{2} f_j^{(\alpha)} + \ldots \doteq e^{\pm \varepsilon \partial_{n_1}} f_j^{(\alpha)};$$

$$f_{n\pm 1}(t;\varepsilon) = \sum_{j=1}^{+\infty} \sum_{\alpha=-j}^{j} \sum_{\rho=max\{1,|\alpha|\}}^{j} \varepsilon^j \left(\mathcal{A}_{j-\rho}^{\pm} f_\rho^{(\alpha)} \right) e^{\mathrm{i}\alpha[\kappa(n\pm 1)-\omega t]};$$

Expansion Operators

1. $\mathcal{A}_\kappa^{\pm} \doteq (\pm\partial_{n_1})^\kappa/\kappa!$: from shift operators as series of derivatives;
2. ∂_{n_1}: derivative operator w. r. t. n_1 (continuos through $\mathcal{C}^{(\infty)}$) with derivatives calculated in $n_1 = \varepsilon n$;

- Similar expansion for the time derivative:

$$\partial_t f_n(t;\varepsilon) = -\mathrm{i}\omega f_n + \sum_{j=2}^{+\infty} \sum_{\alpha=-(j-1)}^{j-1} \sum_{\rho=max\{1,|\alpha|\}}^{j-1} \varepsilon^j \left(\partial_{t_{j-\rho}} f_\rho^{(\alpha)}\right) e^{\mathrm{i}\alpha(\kappa n-\omega t)};$$

The reduced equations

Plug everything into the *dNLS*:

- Order ε:
  $\alpha = 1$: dispersion relation

$$\omega = \frac{1 - \cos(\kappa)}{\sigma^2};$$

  $\alpha = -1$:

$$f_1^{(-1)} = 0;$$

- Order ε^2:
  $\alpha = 1$: group velocity

$$\partial_{t_1} f_1^{(1)} + \frac{\sin(\kappa)}{\sigma^2} \partial_{n_1} f_1^{(1)} = 0, \quad f_1^{(1)} \left(n_1 - \frac{\sin(\kappa)}{\sigma^2} t_1 \right);$$

  $\alpha = 0, -1, \pm 2$:

$$f_1^{(0)} = f_2^{(-1)} = f_2^{(\pm 2)} = 0;$$

- Order ε^3:
  $\alpha = 1$:

$$\partial_{t_1} f_2^{(1)} + \frac{\sin(\kappa)}{\sigma^2} \partial_{n_1} f_2^{(1)} = -\partial_{t_2} f_1^{(1)} + \frac{\mathrm{i}\cos(\kappa)}{2\sigma^2} \partial_{n_1}^2 f_1^{(1)} - \mathrm{i}\rho_2 f_1^{(1)} |f_1^{(1)}|^2,$$

$$\rho_2 \doteq [\beta_1 + (\beta_2 + \beta_3)\cos(\kappa) + \mathrm{i}(\beta_2 - \beta_3)\sin(\kappa)] / N_2.$$

No-secularity conditions

- The right-hand side solves the homogeneous equation: secularity!

$$\partial_{t_1} f_2^{(1)} + \frac{\sin(\kappa)}{\sigma^2} \partial_{n_1} f_2^{(1)} = 0,$$

$$\partial_{t_2} f_1^{(1)} = K_2 \left[f_1^{(1)} \right],$$

$$K_2 \left[f_1^{(1)} \right] \doteq \frac{\mathrm{i}\cos(\kappa)}{2\sigma^2} \partial_{n_1}^2 f_1^{(1)} - \mathrm{i}\rho_2 f_1^{(1)} |f_1^{(1)}|^2 : \textit{NLS equation!}$$

- A_1-integrability condition: $\rho_2 \doteq [\beta_1 + (\beta_2 + \beta_3)] \cos(\kappa) + \mathrm{i}\,(\beta_2 - \beta_3) \sin(\kappa)$ has to be real: it is satisfied iff $\beta_2 = \beta_3$.

Theorem of A_1-integrability:

The dNLS equation

$$\mathrm{i}\dot{u}_n + \frac{1}{2\sigma^2}(u_{n+1} - 2u_n + u_{n-1}) = |u_n|^2 (\beta_1 u_n + \beta_2 u_{n+1} + \beta_3 u_{n-1}) + \\ + |u_n|^4 (\theta_1 u_n + \theta_2 u_{n+1} + \theta_3 u_{n-1}),$$

is A_1-integrable iff $\beta_2 = \beta_3$:

$$\mathrm{i}\dot{u}_n + \frac{1}{2\sigma^2}(u_{n+1} - 2u_n + u_{n-1}) = |u_n|^2 (\beta_1 u_n + \beta_2 [u_{n+1} + u_{n-1}]) + \\ + |u_n|^4 (\theta_1 u_n + \theta_2 u_{n+1} + \theta_3 u_{n-1}),$$

$\alpha = 0,\ -1,\ \pm 2,\ \pm 3$:

$$f_2^{(0)} = f_3^{(-1)} = f_3^{(\pm 2)} = f_3^{(\pm 3)} = 0;$$

- Order ε^4:

  $\alpha = 1$:

$$\partial_{t_1} f_3^{(1)} + \frac{\sin(\kappa)}{\sigma^2} \partial_{n_1} f_3^{(1)} = \mathrm{i}\left(\partial_{t_2} f_2^{(1)} - K_2'\left[f_1^{(1)}\right] f_2^{(1)}\right) + \\ +\mathrm{i}\left(\partial_{t_3} f_1^{(1)} - K_3\left[f_1^{(1)}\right]\right) - \mathrm{i}a|f_1^{(1)}|^2 \partial_{n_1} f_1^{(1)},$$

  $K_3\left[f_1^{(1)}\right]$: flux of first higher order *NLS* symmetry (cmKdV),

  $a \doteq -\beta_1 \tan(\kappa)$;

No-secularity conditions 1

- The right-hand side solves the homogeneous equation: secularity!

$$\partial_{t_1} f_3^{(1)} + \frac{\sin(\kappa)}{\sigma^2} \partial_{n_1} f_3^{(1)} = 0,$$

$$\partial_{t_2} f_2^{(1)} - K_2'\left[f_1^{(1)}\right] f_2^{(1)} = a|f_1^{(1)}|^2 \partial_{n_1} f_1^{(1)} - \left(\partial_{t_3} f_1^{(1)} - K_3\left[f_1^{(1)}\right]\right);$$

$$\partial_{t_2} f_2^{(1)} - K_2' \left[f_1^{(1)} \right] f_2^{(1)} = a |f_1^{(1)}|^2 \partial_{n_1} f_1^{(1)} - \left(\partial_{t_3} f_1^{(1)} - K_3 \left[f_1^{(1)} \right] \right);$$

No-secularity conditions 2

- The red highlighted term on right-hand side solves the homogeneous equation: secularity!

$$\partial_{t_3} f_1^{(1)} = K_3 \left[f_1^{(1)} \right],$$

$$\partial_{t_2} f_2^{(1)} - K_2' \left[f_1^{(1)} \right] f_2^{(1)} = a |f_1^{(1)}|^2 \partial_{n_1} f_1^{(1)};$$

- A_2-integrability conditions: $a \doteq -\beta_1 \tan(\kappa)$ has to be real $\rightarrow$ satisfied!

Theorem of A_2-integrability:

The dNLS equation

$$\mathrm{i}\dot{u}_n + \frac{1}{2\sigma^2}(u_{n+1} - 2u_n + u_{n-1}) = |u_n|^2 (\beta_1 u_n + \beta_2 (u_{n+1} + u_{n-1})) +$$
$$+ |u_n|^4 (\theta_1 u_n + \theta_2 u_{n+1} + \theta_3 u_{n-1}),$$

is A_2-integrable $\forall\ \beta_1,\ \beta_2,\ \theta_i,\ i = 1, 2, 3$;

$\alpha = 0, -1, \pm 2, \pm 3, \pm 4$:

$$f_3^{(0)} = f_4^{(-1)} = f_4^{(\pm 2)} = f_4^{(\pm 3)} = f_4^{(\pm 4)} = 0;$$

- Order ε^5:
  $\alpha = 1$:

No-secularity conditions

$$\partial_{t_1} f_4^{(1)} + \frac{\sin(\kappa)}{\sigma^2} \partial_{n_1} f_4^{(1)} = 0,$$

$$\partial_{t_2} f_3^{(1)} - K_2' \left[f_1^{(1)} \right] f_3^{(1)} = g_2(3): \text{ forced linearized } NLS,$$

$$\partial_{t_3} f_2^{(1)} - K_3' \left[f_1^{(1)} \right] f_2^{(1)} = g_3(2): \text{ forced linearized } cmKdV,$$

$$\partial_{t_4} f_1^{(1)} = K_4 \left[f_1^{(1)} \right]: \text{ flux of second higher order } NLS \text{ symmetry};$$

- A_3-integrability conditions (on the coefficient of $g_2(3)$):
  $\beta_1 = \theta_1 = \theta_2 = \theta_3 = 0 \rightarrow$ Ablowitz-Ladik!;

Theorem of A_3-integrability:

The only dNLS belonging to our class

$$\mathrm{i}\dot{u}_n + \frac{1}{2\sigma^2}(u_{n+1} - 2u_n + u_{n-1}) = |u_n|^2(\beta_1 u_n + \beta_2 u_{n+1} + \beta_3 u_{n-1}) + + |u_n|^4(\theta_1 u_n + \theta_2 u_{n+1} + \theta_3 u_{n-1}),$$

being A_3-integrable, is the Ablowitz-Ladik dNLS equation

$$\mathrm{i}\partial_t u_n(t) + \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{2\sigma^2} = \beta_2 |u_n(t)|^2 (u_{n+1}(t) + u_{n-1}(t)).$$

Other partial difference equations

- offcentrically discretized *KdV* equation: A_0-asymptotically integrable!

$$u_2 - u_{-2} = \frac{\alpha}{4}\left[u_{111} - 3u_1 + 3u_{-1} - u_{-1-1-1}\right] - \frac{b}{2}\left[u_1^2 - u^2\right];$$

- symmetrically discretized *KdV* equation: A_2-asymptotically integrable!

$$u_2 - u_{-2} = \frac{\alpha}{4}\left[u_{111} - 3u_1 + 3u_{-1} - u_{-1-1-1}\right] - \frac{b}{2}\left[u_1^2 - u_{-1}^2\right];$$

- Zabusky-Kruskal *KdV*

$$\dot{q}_n = \frac{1}{2h^3}\left(q_{n+2} - 2q_{n+1} + 2q_{n-1} - q_{n-2}\right) + \frac{1}{h}\left(q_{n+1} + q_n + q_{n-1}\right)\left(q_{n+1} - q_{n-1}\right)$$

  A_2-asymptotically integrable!

- *lpKdV* equation: A_3

$$\alpha\left(u_{n+1,m+1} - u_{n,m}\right) + \beta\left(u_{n+1,m} - u_{n,m+1}\right) - \\ - \left(u_{n+1,m} - u_{n,m+1}\right)\left(u_{n+1,m+1} - u_{n,m}\right) = 0;$$

- *Hietarinta* equation (A_1: linearizable $\rightarrow A_\infty$).

$$\frac{u_{n,m} + e_2}{u_{n,m} + e_1} \cdot \frac{u_{n+1,m+1} + o_2}{u_{n+1,m+1} + o_1} = \frac{u_{n+1,m} + e_2}{u_{n+1,m} + o_1} \cdot \frac{u_{n,m+1} + o_2}{u_{n,m+1} + e_1}.$$

Classification of lattice equations on the square

- Dispersive affine linear equation on the square:

$$
\begin{aligned}
a_1(u_{n,m} + u_{n+1,m+1}) + a_2(u_{n+1,m} + u_{n,m+1}) + \\
&+(\alpha_1 - \alpha_2)\, u_{n,m}u_{n+1,m} + (\alpha_1 + \alpha_2)\, u_{n,m+1}u_{n+1,m+1} + \\
&+ (\beta_1 - \beta_2)\, u_{n,m}u_{n,m+1} + (\beta_1 + \beta_2)\, u_{n+1,m}u_{n+1,m+1} + \\
&+ \gamma_1 u_{n,m}u_{n+1,m+1} + \gamma_2 u_{n+1,m}u_{n,m+1} + \\
&+ (\xi_1 - \xi_3)\, u_{n,m}u_{n+1,m}u_{n,m+1} + (\xi_1 + \xi_3)\, u_{n,m}u_{n+1,m}u_{n+1,m+1} + \\
&+ (\xi_2 - \xi_4)\, u_{n+1,m}u_{n,m+1}u_{n+1,m+1} + (\xi_2 + \xi_4)\, u_{n,m}u_{n,m+1}u_{n+1,m+1} + \\
&+ \zeta u_{n,m}u_{n+1,m}u_{n,m+1}u_{n+1,m+1} = 0,
\end{aligned}
$$

  a_1, a_2, α_1, α_2, β_1, β_2, γ_1, γ_2, ξ_1, ξ_2, ξ_3, ξ_4, ζ real parameters and $|a_1| \neq |a_2|$.

- Linear dispersion relation:

$$
\omega(\kappa) = \arctan\left[\frac{(a_1^2 - a_2^2)\sin(\kappa)}{(a_1^2 + a_2^2)\cos(\kappa) + 2a_1a_2}\right];
$$

Theorem of A_1-integrability: *The only A_1-integrable eq. in our class are characterized by:*

- *Case 1*:

$$\begin{cases} 2a_1a_2\alpha_1 = \gamma_1 a_2^2 + \gamma_2 a_1^2, \\ 2a_1a_2(a_1-a_2)\beta_1 = (a_1+a_2)(\gamma_2 a_1^2 - \gamma_1 a_2^2), \\ (a_2+a_1)\beta_2 = (a_2-a_1)\alpha_2, \\ (a_2^2-a_1^2)(\xi_1-\xi_2) = [\gamma_1(a_1-3a_2)-\gamma_2(a_2-3a_1)]\,\alpha_2, \\ (a_1+a_2)(\xi_3-\xi_4) = (\gamma_2-\gamma_1)\alpha_2. \end{cases} \tag{10}$$

- *Case 2*:

$$\begin{cases} 2a_1a_2(a_1-a_2)\alpha_1 = (a_1+a_2)(\gamma_2 a_1^2 - \gamma_1 a_2^2), \\ 2a_1a_2\beta_1 = \gamma_1 a_2^2 + \gamma_2 a_1^2, \\ (a_2-a_1)\beta_2 = (a_2+a_1)\alpha_2, \\ (a_2-a_1)(\xi_1-\xi_2) = (\gamma_1-\gamma_2)\alpha_2, \\ (a_2-a_1)^2(\xi_3-\xi_4) = [\gamma_2(a_2-3a_1)-\gamma_1(a_1-3a_2)]\,\alpha_2. \end{cases} \tag{11}$$

Theorem of A_1-integrability: (cont.)

- *Case 3*:

$$\begin{cases} \gamma_1 a_2 = \gamma_2 a_1, \\ \alpha_1 = \beta_1 = \frac{1}{2}(\gamma_1 + \gamma_2), \\ a_1(\xi_1 - \xi_2) = -\alpha_2 \gamma_1, \\ a_1(\xi_3 - \xi_4) = \beta_2 \gamma_1. \end{cases} \tag{12}$$

- *Case 4*:

$$\begin{cases} \alpha_2 = \beta_2 = 0, \\ \xi_1 = \xi_2, \\ \xi_3 = \xi_4. \end{cases} \tag{13}$$

Theorem of A_1-integrability: (cont.)

- *Case 5*:

$$\begin{cases} a_2 = 2a_1, \\ \alpha_1 = \beta_1, \\ \alpha_2 = -\beta_2, \\ \gamma_2 = 2\gamma_1, \\ a_1(\xi_1 - \xi_2) = a_1(\xi_3 - \xi_4) = -\alpha_2\gamma_1. \end{cases} \tag{14}$$

- *Case 6*:

$$\begin{cases} a_1 = 2a_2, \\ \alpha_1 = \beta_1, \\ \alpha_2 = \beta_2, \\ \gamma_1 = 2\gamma_2, \\ a_1(\xi_1 - \xi_2) = -a_1(\xi_3 - \xi_4) = -\alpha_2\gamma_1. \end{cases} \tag{15}$$

Conclusions

1. *Integrability test* suitable for a large variety of nonlinear systems;
2. We have shown that among a class of *dNLS* equations considered in the literature only the Ablowitz-Ladik *dNLS* is integrable;
3. A_1-classification of dispersive affine linear equation on the square.

Open problems

- What happens if we do not require the $\mathcal{C}^{(\infty)}$ property of solutions: can we still get *discrete integrable systems*;
- Extend to other discrete systems as *weakly dissipative* systems: *Burgers* hierarchy;
- Find the appropriate normal form theory for discrete equations;

Conclusions

Open problems

- In the A_1-classification of dispersive affine linear equation on the square one equation emerges as a possibly integrable equation:

$$
\begin{aligned}
a_1[u_{n,m} + u_{n+1,m+1} + 2(u_{n+1,m} + u_{n,m+1})] + \\
+3\gamma_1 [u_{n,m}u_{n+1,m} + u_{n,m+1}u_{n+1,m+1} + u_{n,m}u_{n,m+1} + u_{n+1,m}u_{n+1,m+1}] \\
+ \gamma_1[u_{n,m}u_{n+1,m+1} + 2u_{n+1,m}u_{n,m+1}] + \\
+ (\xi_1 - \xi_3) [u_{n,m}u_{n+1,m}u_{n,m+1} + u_{n+1,m}u_{n,m+1}u_{n+1,m+1}] + \\
+ (\xi_1 + \xi_3) [u_{n,m}u_{n+1,m}u_{n+1,m+1} + u_{n,m}u_{n,m+1}u_{n+1,m+1}] + \\
+ \zeta u_{n,m}u_{n+1,m}u_{n,m+1}u_{n+1,m+1} = 0,
\end{aligned}
$$

  Analyze its A_3 integrability.
- Integrability test for *maps*;
- Dependence of degree of asymptotic integrability from the solutions used.

THANK YOU