Definitions and Predictions of Integrability for Difference Equations

Jarmo Hietarinta

Department of Physics, University of Turku, FIN-20014 Turku, Finland

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UNIVERSITY OF TURKU

Why discrete?

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- Perhaps discrete things are more fundamental than continuous
- Many mathematical constructs can be interpreted as difference relations, e.g., recursion relations.
- Need to discretize continuous equations for numerical analysis
- Interesting mathematics in the background, e.g., elliptic functions.
- Continuum integrability is well established, all easy things have already been done. Discrete integrability relatively new, still new things to be discovered.

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Preannoucement: SIDE in Beijing 2012.

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More detailed questions:

- Can we say anything about x_n without actually computing every intermediate step?
- Can we find formulae like $x_n = \phi(x_0, x_1; n)$ where ϕ is some reasonable function?
- How does the error in the initial values propagate? Does the resulting ambiguity grow as n^2, or as 2^n?

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In these lectures: we take a look on various meanings of integrability for difference equations, and the possible associated algorithmic methods to identify (partial) integrability.

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Sensitive dependence on the initial value:

$$\frac{dy_n}{dc_0} = \tfrac{1}{2}\, 2^n \sin(2^n c_0)$$

Thus error grows exponentially: “chaotic”.

Examples and continuum limits

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Let us take the continuum limit: set

$$\epsilon n = z,\, x_n = f(z),\, x_{n\pm 1} = f(z \pm \epsilon), \quad \epsilon \to 0,\, n \to \infty,\, \epsilon n \text{ fixed}$$

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This yields

$$3f + \epsilon^2 f'' = \frac{\alpha + \beta z/\epsilon}{f} + b.$$

The get rid of the denominator we must take

$$f(z) = c_1 + c_2 \epsilon^\kappa y(z),$$

and expand. The power $\kappa > 0$ is to determined.

$$3c_1+3c_2\epsilon^{\kappa}y(z)+3c_2\epsilon^{2+\kappa}y'' = b+\frac{1}{c_1}(\alpha+\beta z/\epsilon)(1-\frac{c_2}{c_1}\epsilon^{\kappa}y+(\frac{c_2}{c_1})^2\epsilon^{2\kappa}y^2\dots$$

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To balance terms we must take $\kappa = 2$, β high order in ϵ, then

ϵ^0: $3c_1 = b + \alpha/c_1$
ϵ^2: $3c_2 = -c_2\alpha/c_1^2$

leading to

$$c_1 = \frac{b}{6}, \quad \alpha = -\frac{b^2}{12}$$

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Finally at ϵ^4 we get the first Painleve equation

$$y'' = 6y^2 + z,$$

if we take

$$c_2 = -\frac{b}{3}, \quad \beta = -\frac{b^2}{18}\epsilon^5.$$

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- Local analysis (for complex time) to check whether solutions have movable singularities (Painlevé method). [Search program by Painlevé, Gambier, etc.]
- Growth analysis of the solution (Nevanlinna theory)

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What about difference equations?

Maybe for a discrete Painlevé test we should again study what happens at a singularity.

What about growth analysis?
Recall that difference equations can trivially be solved step by step, what is the growth of the resulting expression?

Singularity analysis for difference equations

Grammaticos, Ramani, and Papageorgiou, [*Phys. Rev. Lett.* 67 (1991) 1825] proposed The Singularity Confinement Criterion as an analogue of the Painleve test.

Idea: If the dynamics leads to a singularity then after a few steps one should be able to get out of it (confinement), and this should take place *without loss of information.*
(in contrast: attractors absorb information)

This amounts to the requirement of well defined evolution even near singular points.

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Using this principle it has been possible to find discrete analogies of Painlevé equations. [Ramani, Grammaticos and JH, *Phys. Rev. Lett.* 67 (1991) 1829, and many others]

Singularity confinement in practice

Consider first the autonomous case of dPI

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The sequence continues as:

$$\begin{aligned} x_1 &= -0 - \mathbf{u} + a/0 + b = \infty, \\ x_2 &= -\infty - 0 + a/\infty + b = -\infty, \\ x_3 &= +\infty - \infty - a/\infty + b = \ ? \end{aligned}$$

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To resolve "$\infty - \infty$":
assume $x_0 = \epsilon$ (small) and redo the calculations.

Detailed singularity confinement calculation

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The singularity pattern is $\ldots, 0, \infty, -\infty, 0, \ldots$.

Non-confined singularity

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In general

$$x_k = k\,\frac{a}{\epsilon} + \dots,$$

and the singularity is not confined, ever.
Furthermore: there are no ambiguities.

The success of singularity confinement

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Problem: x_4 should start like $\mathbf{u} + \dots$!

$$x_4 = -\frac{a_3 - a_2 - a_1 + a_0}{a_2 + a_1 + a_0} \frac{a_0}{\epsilon} + \dots$$

x_4 should start like $\mathbf{u} + \dots \Longrightarrow$

The condition for singularity confinement at this same step is:

$$a_{n+3} - a_{n+2} - a_{n+1} + a_n = 0, \; \forall n$$

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with solution

$$a_n = \alpha + \beta n + \gamma (-1)^n. \qquad (*)$$

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Recall the form of the discrete Painlevé equation (d-PI)

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$$x_{n+1} + x_n + x_{n-1} = \frac{\alpha + \beta n}{x_n} + b.$$

In general, with a_n as in (*) the singularity is confined, and

$$x_4 := \frac{\mathbf{u}(\alpha + \gamma) + 2b\beta}{\alpha + 3\beta - \gamma} + O(\epsilon),$$

in particular, if $\beta = \gamma = 0$ (i.e., $a_n = \alpha$), $x_4 = \mathbf{u} + \dots$

Singularity confinement in projective space

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Write it as a first order system

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Then homogenize by substituting $x_n = u_n/f_n$, $y_n = v_n/f_n$:

$$\begin{cases} \frac{u_{n+1}}{f_{n+1}} &= -\frac{u_n}{f_n} - \frac{v_n}{f_n} + a_n \frac{f_n}{u_n} + b, \\ \frac{v_{n+1}}{f_{n+1}} &= \frac{u_n}{f_n}, \end{cases}$$

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Then clearing denominators yields a polynomial map in $\mathbb{P}^2$

$$\begin{cases} u_{n+1} &= -u_n(u_n + v_n) + f_n(a_n f_n + b u_n), \\ v_{n+1} &= u_n^2, \\ f_{n+1} &= f_n u_n. \end{cases}$$

Note: default growth of degree (= complexity): $\deg(u_n) = 2^n$

The sequence that led to a singularity was
$x_{-1} = \mathbf{u}$, $x_0 = 0$, $x_1 = \infty$, $x_2 = \infty$, $x_3 = \infty - \infty =$?

The sequence that led to a singularity was
$x_{-1} = \mathbf{u},\ x_0 = 0,\ x_1 = \infty,\ x_2 = \infty,\ x_3 = \infty - \infty = \ ?$

In projective space we have

$$\begin{pmatrix} 0 \\ \mathbf{u} \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

The last term is a true singularity, since it is not in $\mathbb{P}^2$.

For the detailed ϵ study with $x_{-1} = \mathbf{u}$, $x_0 = \epsilon$ we have

$$\begin{pmatrix} x_0 \\ x_{-1} \\ 1 \end{pmatrix} \approx \begin{pmatrix} u_0 \\ v_0 \\ f_0 \end{pmatrix} = \begin{pmatrix} \epsilon \\ \mathbf{u} \\ 1 \end{pmatrix},$$

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$$\begin{pmatrix} x_1 \\ x_0 \\ 1 \end{pmatrix} \approx \begin{pmatrix} u_1 \\ v_1 \\ f_1 \end{pmatrix} = \begin{pmatrix} a_0 + (-\mathbf{u} + b)\epsilon + \dots \\ \epsilon^2 \\ \epsilon \end{pmatrix}.$$

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$$\begin{pmatrix} x_2 \\ x_1 \\ 1 \end{pmatrix} \approx \begin{pmatrix} u_2 \\ v_2 \\ f_2 \end{pmatrix} = \begin{pmatrix} -a_0^2 + \epsilon a_0(2\mathbf{u} - b) + \dots \\ a_0^2 + 2\epsilon a_0(-\mathbf{u} + b) + \dots \\ \epsilon a_0 + \epsilon^2(-\mathbf{u} + b) + \dots \end{pmatrix}.$$

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$$\begin{pmatrix} x_3 \\ x_2 \\ 1 \end{pmatrix} \approx \begin{pmatrix} u_3 \\ v_3 \\ f_3 \end{pmatrix} = \begin{pmatrix} \epsilon^2 a_0^2(-a_0 + a_1 + a_2) + \dots \\ a_0^4 + 2\epsilon a_0^3(-2\mathbf{u} + b) \dots \\ -\epsilon a_0^3 + \epsilon^2 a_0^2(3\mathbf{u} - 2b) + \dots \end{pmatrix}$$

$$\begin{pmatrix} u_4 \\ v_4 \\ f_4 \end{pmatrix} = \begin{pmatrix} \epsilon^2 a_0^6 A_3 + \epsilon^3 a_0^5 (b(4A_3 + a_0 - a_2) - \mathbf{u}(6A_3 + a_0)) + \dots \\ \epsilon^4 a_0^4 A_2^2 + \dots \\ -\epsilon^3 a_0^5 A_2 + \dots \end{pmatrix}$$

$(A_2 = a_2 + a_1 - a_0, A_3 = a_0 - a_1 - a_2 + a_3.)$

This is the crucial point of singularity confinement.

$$\begin{pmatrix} u_4 \\ v_4 \\ f_4 \end{pmatrix} = \begin{pmatrix} \epsilon^2 a_0^6 A_3 + \epsilon^3 a_0^5 (b(4A_3 + a_0 - a_2) - \mathbf{u}(6A_3 + a_0)) + \dots \\ \epsilon^4 a_0^4 A_2^2 + \dots \\ -\epsilon^3 a_0^5 A_2 + \dots \end{pmatrix}$$

$(A_2 = a_2 + a_1 - a_0, A_3 = a_0 - a_1 - a_2 + a_3.)$

This is the crucial point of singularity confinement.

If $A_3 = 0, A_2 \neq 0$ then ϵ^3 is a common factor and can be divided out and then the $\epsilon \to 0$ limit yields

$$\begin{pmatrix} u_4 \\ v_4 \\ f_4 \end{pmatrix} = \begin{pmatrix} (a_0(\mathbf{u} - b) + a_2 b) \\ 0 \\ a_3 \end{pmatrix}.$$

Thus we have emerged from the singularity and in particular recovered the initial data **u**.

- The cancellation of the common factor ϵ^3 removes the singularity.
- Any cancellation also reduces growth of complexity, as defined by the degree of the iterate.

These are two sides of the same phenomenon.

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The precise amount of cancellation will be crucial.

- growth is linear in $n \Rightarrow$ equation is linearizable.
- growth is polynomial in $n \Rightarrow$ equation is integrable.
- growth is exponential in $n \Rightarrow$ equation is chaotic.

Singularity confinement is not sufficient

Counterexample (JH and C Viallet, PRL 81, 325 (1999))

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^2}.$$

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Epsilon analysis of singularity confinement:
Assume $x_{-1} = \mathbf{u}$, $x_0 = \epsilon$ and then

$$\begin{aligned} x_1 &= \epsilon^{-2} - \mathbf{u} + \epsilon, \\ x_2 &= \epsilon^{-2} - \mathbf{u} + \epsilon^4 + O(\epsilon^6), \\ x_3 &= -\epsilon + 2\epsilon^4 + O(\epsilon^6), \\ x_4 &= \mathbf{u} + 3\epsilon + O(\epsilon^3), \end{aligned}$$

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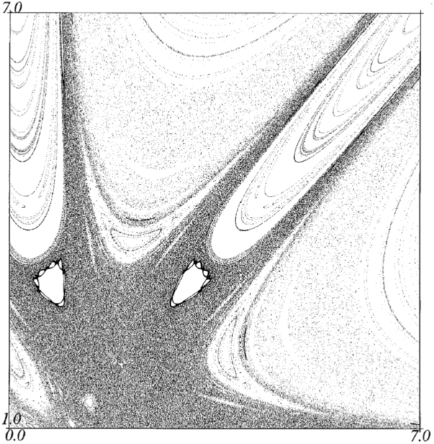
Thus singularity is confined with pattern $\dots, 0, \infty, \infty, 0, \dots$.
Furthermore, the initial information $\mathbf{u}$ is recovered in x_4. OK?

No! The HV map shows numerical chaos

$$x_{n+1} + x_{n-1} = x_n + \frac{7}{x_n^2}$$



Singularity confinement $\Rightarrow$ cancellations $\Rightarrow$ reduced growth of complexity.

Singularity confinement ⇒ cancellations ⇒ reduced growth of complexity.

Reduction must be strong enough!

For the previous chaotic model the degrees grow as

$$1, 3, 9, 27, 73, 195, 513, 1347, 3529, \dots$$

which grows asymptotically as $d_n \approx [(3+\sqrt{5})/2]^n$.

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For the previous Painlevé equation the degrees grow as

$$1, 2, 4, 8, 13, 20, 28, 38, 49, 62, 76, \dots$$

which is fitted by $d_n = \frac{1}{8}(9 + 6n^2 - (-1)^n)$. [JH and Viallet, Chaos, Solitons and Fractals, **11**, 29-32 (2000).]

Summary

- Singularity confinement is necessary for a well defined evolution
- Easy to verify
- Can be used effectively for de-autonomizing a given map
- Not sufficient for integrable evolution

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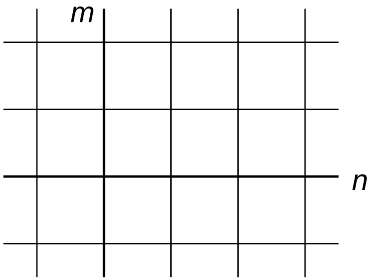
Improvements / other tests

- Require slow growth of complexity (Veselov, Arnold, Falqui and Viallet)
- Consider the map over finite fields and study its orbit statistics (Roberts and Vivaldi)
- Nevanlinna theory for difference equations. (Halburd et al)
- Diophantine integrability (numerically fast) (Halburd)

Dynamics in a square lattice

The basic setting: an infinite rectangular lattice in the plane:



Values of the dynamical variable u given at intersections, $u_{n,m}$.

Examples

The discrete KdV can be given as

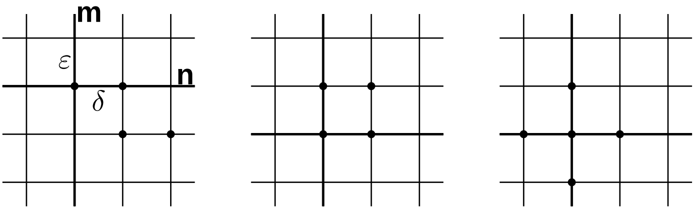
$$\alpha(y_{n+2,m-1} - y_{n,m}) = \left(\frac{1}{y_{n+1,m-1}} - \frac{1}{y_{n+1,m}}\right)$$

or in the "potential" form

$$(u_{n,m+1} - u_{n+1,m})(u_{n,m} - u_{n+1,m+1}) = p^2 - q^2$$

The equation of "similarity constraint" for KdV is given by

$$(\lambda(-1)^{n+m} + \tfrac{1}{2})u_{n,m} + \frac{np^2}{u_{n-1,m} - u_{n+1,m}} + \frac{mq^2}{u_{n,m-1} - u_{n,m+1}} = 0$$



KdV in applications

Several numerical acceleration algorithms (for partial sums) are integrable lattice equations.

The Shanks-Wynn ϵ-algorithm: Assume the initial sequences $\epsilon_0^{(m)} = 0$, $\epsilon_1^{(m)} = S_m$, and generate new sequences $\epsilon_n^{(m)}$ (that approach the limit S_∞ faster) by

$$(\epsilon_{n+1}^{(m)} - \epsilon_{n-1}^{(m+1)})(\epsilon_n^{(m+1)} - \epsilon_n^{(m)}) = 1.$$

This is the integrable discrete potential KdV equation.

Similarly, Bauer's η-algorithm ($X_k^{(m)} = [\eta_k^{(m)}]^{(-1)^{k+1}}$)

$$X_{n+1}^{(m)} - X_{n-1}^{(m+1)} = \frac{1}{X_n^{(m+1)}} - \frac{1}{X_n^{(m)}}$$

is the integrable discrete KdV equation.

Relationship between dKdV and dpKdV

Let $y_{n,m} = 1 + W_{n+m,m+1}$ then dKdV becomes

$$\alpha(W_{n,m+1} - W_{n+1,m}) = \frac{1}{1+W_{n,m}} - \frac{1}{1+W_{n+1,m+1}}$$

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Next let $W_{n,m} = (U_{n-1,m-1} - U_{n,m})/(p+q)$, which implies

$$\frac{\alpha}{p+q}(U_{n-1,m} - U_{n,m+1} - U_{n,m-1} + U_{n+1,m}) = \frac{1}{1 + \frac{U_{n-1,m-1} - U_{n,m}}{p+q}} - \frac{1}{1 + \frac{U_{n,m} - U_{n+1,m+1}}{p+q}}.$$

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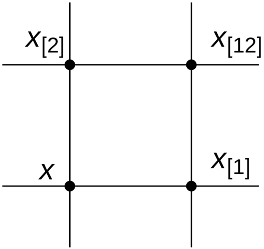
The red part is a double shift or the blue part, separate as

$$1 + \frac{U_{n,m+1} - U_{n+1,m}}{p-q} = \frac{1}{1+\frac{U_{n,m}-U_{n+1,m+1}}{p+q}},$$

where $\alpha = (p+q)/(p-q)$ and the separation constant = 1.
This is the dpKdV.

Closer look at quadrilateral lattices

$$x_{n,m} = x_{00} = x$$
$$x_{n+1,m} = x_{10} = x_{[1]} = \widetilde{x}$$
$$x_{n,m+1} = x_{01} = x_{[2]} = \widehat{x}$$
$$x_{n+1,m+1} = x_{11} = x_{[12]} = \widehat{\widetilde{x}}$$



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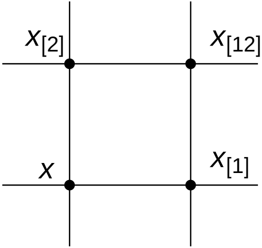
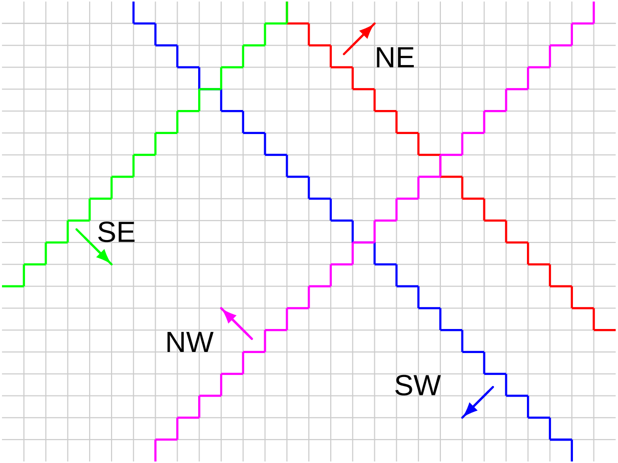


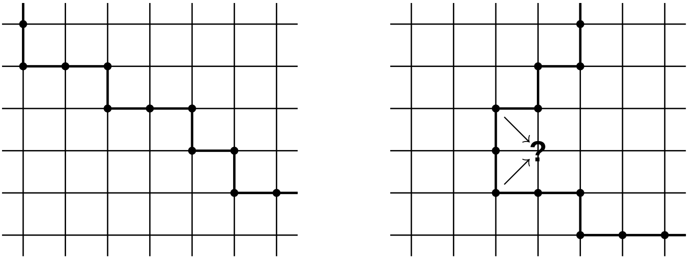
The four corner values are related by a multi-linear equation:

$$k\, x x_{[1]} x_{[2]} x_{[12]} + l_1\, x x_{[1]} x_{[2]} + l_2\, x x_{[1]} x_{[12]} + l_3\, x x_{[2]} x_{[12]} + l_4\, x_{[1]} x_{[2]} x_{[12]}$$
$$+ s_1\, x x_{[1]} + s_2\, x_{[1]} x_{[2]} + s_3\, x_{[2]} x_{[12]} + s_4\, x_{[12]} x + s_5\, x x_{[2]} + s_6\, x_{[1]} x_{[12]}$$
$$+ q_1\, x + q_2\, x_{[1]} + q_3\, x_{[2]} + q_4\, x_{[12]} + u \equiv Q(x, x_{[1]}, x_{[2]}, x_{[12]}; p_1, p_2) = 0$$

The p_i are some parameters associated with shift directions $[i]$, they may appear in the coefficients k, l_i, s_i, q_i, u.

This definition allows well-defined evolution from any staircase-like initial condition, up or down.





Steplike initial values OK.
Any overhang would lead into trouble.

Further examples

Lattice (potential) KdV

$$(p_1 - p_2 + x_{n,m+1} - x_{n+1,m})(p_1 + p_2 + x_{n,m} - x_{n+1,m+1}) = p_1^2 - p_2^2,$$

or after translation $x_{n,m} = u_{n,m} + p_1 n + p_2 m$

$$(u_{n,m+1} - u_{n+1,m})(u_{n,m} - u_{n+1,m+1}) = p_1^2 - p_2^2,$$

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Lattice MKdV

$$p_1(x_{n,m}x_{n,m+1} - x_{n+1,m}x_{n+1,m+1}) = p_2(x_{n,m}x_{n+1,m} - x_{n,m+1}x_{n+1,m+1}),$$

Further examples

Lattice (potential) KdV

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$$p_1(x_{n,m}x_{n,m+1} - x_{n+1,m}x_{n+1,m+1}) = p_2(x_{n,m}x_{n+1,m} - x_{n,m+1}x_{n+1,m+1}),$$

Lattice SKdV

$$(x - \widetilde{x})(\widehat{x} - \widehat{\widetilde{x}})p_2^2 = (x - \widehat{x})(\widetilde{x} - \widehat{\widetilde{x}})p_1^2.$$

Continuum limit

The famous Korteweg-de Vries equation in potential form is

$$v_t = v_{xxx} + 3v_x^2,$$

how is this related to the dpKdV given by

$$(p - q + u_{n,m+1} - u_{n+1,m})(p + q + u_{n,m} - u_{n+1,m+1}) = p^2 - q^2$$

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In the “straight” continuum limit we take

$$u(n, m+k) = y_n(\xi + \epsilon k), \quad q = 1/\epsilon$$

and expand, obtaining in leading order

$$\partial_\xi(y_n + y_{n+1}) = 2p(y_{n+1} - y_n) - (y_{n+1} - y_n)^2$$

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$$u_{n,m+1} = w_N(\tau + \epsilon), \quad u_{n+1,m+1} = w_{N+1}(\tau + \epsilon)$$

and then expand in ϵ. The result is (at order ϵ)

$$\partial_\tau w_N = \frac{2p}{2p + w_{N-1} - w_{N+1}} - 1.$$

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$$\partial_\tau w_N = \frac{2p}{2p + w_{N-1} - w_{N+1}} - 1.$$

If we let $W_n = 2p + w_{N-2} - w_N$ then we get

$$\dot{W}_n = 2p\left(\frac{1}{W_{N+1}} - \frac{1}{W_{N-1}}\right)$$

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Next we expand $y_{n+k} = v(\tau + k\epsilon)$ in ϵ, with $p = 1/\epsilon$, and obtain

$$2v_\xi + \epsilon v_{\xi\tau} + \tfrac{1}{2}\epsilon^2 v_{\xi\tau\tau} \cdots = 2v_\tau + \epsilon v_{\tau\tau} + \tfrac{1}{3}\epsilon^2 v_{\tau\tau\tau} + \epsilon^2 v_\tau^2 + \dots$$

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Now we need to redefine the independent variables from ξ, τ to x, t using

$$\partial_\tau = \partial_x + \tfrac{1}{12}\epsilon^2 \partial_t, \quad \partial_\xi = \partial_x$$

and then we get

$$v_t = v_{xxx} + 6v_x^2$$

which is the potential form of KdV. [$v_x = u$]

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Next take a continuum limit in N by

$$w_{N+k} = v(x + k\epsilon),\, p = 2/\epsilon$$

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leading to

$$2v_\tau - (\epsilon^2 v_x + \tfrac{1}{6}\epsilon^4 v_{xxx})(v_\tau + 1) + \cdots = 0.$$

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leading to

$$2v_\tau - (\epsilon^2 v_x + \tfrac{1}{6}\epsilon^4 v_{xxx})(v_\tau + 1) + \cdots = 0.$$

As before we need to change “time”, now by

$$\partial_\tau = \tfrac{1}{2}\epsilon^2 \partial_x + \tfrac{1}{12}\epsilon^4 \partial_t.$$

Then at the lowest nontrivial order (ϵ^4) we find

$$v_t = v_{xxx} + 3v_x^2.$$

Singularity confinement in 2D

Grammaticos, Ramani, Papageorgiou, PRL **67**, 1825 (1991)
As an example let us consider dKdV

$$w_{n+1,m+1} = w_{n,m} + \frac{1}{w_{n+1,m}} - \frac{1}{w_{n,m+1}}.$$

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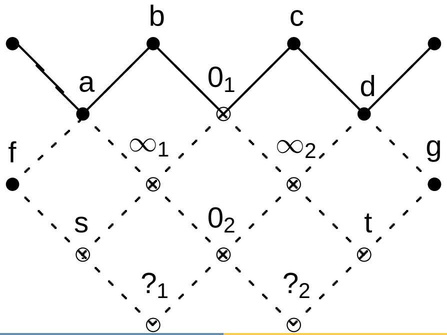
$$w_{n+1,m+1} = w_{n,m} + \frac{1}{w_{n+1,m}} - \frac{1}{w_{n,m+1}}.$$

The initial data is $a, b, 0, c, d, f, g$.



A more detailed analysis with the initial value $0_1 = \varepsilon$ (small) yields the following values at the subsequent iterations

$$\infty_1 = b + \frac{1}{\varepsilon} - \frac{1}{a}, \quad \infty_2 = c + \frac{1}{d} - \frac{1}{\varepsilon},$$

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at the first stage, and on the next

$$s = a + \frac{1}{\infty_1} - \frac{1}{f}, \quad t = d + \frac{1}{g} - \frac{1}{\infty_2},$$

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Then at the next step we can resolve the ambiguities:

$$\begin{aligned} ?_1 &= \infty_1 + \tfrac{1}{0_2} - \tfrac{1}{s} = c + \tfrac{1}{d} - \tfrac{1}{a-1/f} + \mathcal{O}(\varepsilon) \\ ?_2 &= \infty_2 + \tfrac{1}{t} - \tfrac{1}{0_2} = b - \tfrac{1}{a} + \tfrac{1}{d+1/g} + \mathcal{O}(\varepsilon) \end{aligned}$$

Thus the singularity is confined.

Algebraic entropy study for lattices?

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- Growth of complexity (=degree of iterate) is usually exponential.
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What about growth analysis for lattices?

The setting

Consider a quadratic map in a quadrilateral lattice.

$$p_1\, x x_{[1]} + p_2\, x_{[1]} x_{[2]} + p_3\, x_{[2]} x_{[12]} + p_4\, x_{[12]} x + p_5\, x x_{[2]} + p_6\, x_{[1]} x_{[12]}$$
$$+ q_1\, x + q_2\, x_{[1]} + q_3\, x_{[2]} + q_4\, x_{[12]} + u = 0$$

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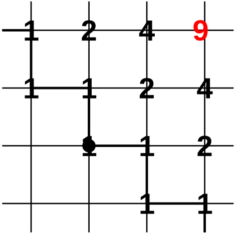
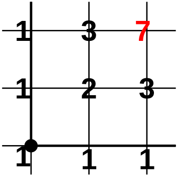
$$+ q_1\, x + q_2\, x_{[1]} + q_3\, x_{[2]} + q_4\, x_{[12]} + u = 0$$

Write the map in the projective plane with $x = v/f$:

$$\begin{cases} v_{[12]} & = & p_1\, v\, v_{[1]} f_{[2]} + p_2\, v_{[1]} v_{[2]} f + p_5\, v\, v_{[2]} f_{[1]} \\ & & + q_1\, v\, f_{[1]} f_{[2]} + q_2\, v_{[1]} f_{[2]} f + q_3\, v_{[2]} f_{[1]} f + u\, f\, f_{[1]} f_{[2]}, \\ f_{[12]} & = & p_3\, v_{[2]} f_{[1]} f + p_4\, v\, f_{[1]} f_{[2]} + p_6\, v_{[1]} f_{[2]} f + q_4 f\, f_{[1]} f_{[2]}. \end{cases}$$

Default degree growth in a staircase and in a corner:





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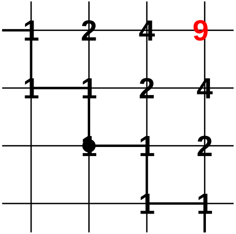
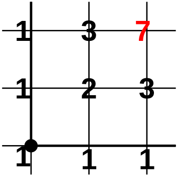
On those points v is free, but f’s should be the same.

Default degree growth in a staircase and in a corner:





Initial values given on the points marked with “1”.
On those points v is free, but f's should be the same.
Default degree growth:

$$\deg(z_{[12]}) = \deg(z) + \deg(z_{[1]}) + \deg(z_{[2]}) - 1,$$

($z = v$ or f, they have the same degree).
The extra -1 is because the map is quadratic and a common f is cancelled.

Interesting factorization takes place at degree 9 or 7.

Default asymptotic growth for the staircase: $\frac{1}{2}(1+\sqrt{2})^n$.

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What happens with well known models? [Tremblay, Grammaticos and Ramani, Phys. Lett. A **278** 319 (2001).]
For dpKdV they obtain degrees

$$
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
1 & 4 & 7 & 10 & 13 & 16 & \dots \\
1 & 3 & 5 & 7 & 9 & 11 & \dots \\
1 & 2 & 3 & 4 & 5 & 6 & \dots \\
1 & 1 & 1 & 1 & 1 & 1 & \dots
\end{array}
\qquad
\begin{array}{ccccccc}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
1 & 2 & 4 & 7 & 11 & 16 & \dots \\
1 & 1 & 2 & 4 & 7 & 11 & \dots \\
 & 1 & 1 & 2 & 4 & 7 & \dots \\
 & & 1 & 1 & 2 & 4 & \dots \\
 & & & \ddots & \ddots & \ddots &
\end{array}
$$

In the corner case $d_{nm} = nm + 1$, in the staircase $d_N = 1 + N(N-1)/2$. Polynomial growth.

Cancelling factors

KdV:

$$(x_{n,m+1} - x_{n+1,m})(x_{n,m} - x_{n+1,m+1}) = a,$$

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"Stair" at (2,2) (maximal degree 9)

$$v_{22}, f_{22} = (\text{main part of degree 7}) \times (v_{01} - v_{10})^2.$$

"Corner" at (2,2) (maximal degree 7)

$$v_{22}, f_{22} = (\text{main part of degree 5}) \times (v_{01} - v_{10})^2.$$

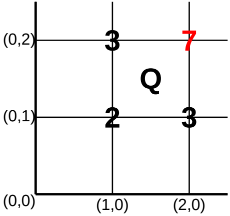
where z is v or f. The main parts of v and f are different, therefore in each case $GCD(v_{22}, f_{22}) = (v_{01} - v_{10})^2$.

Search based on factorization

Integrable maps seem to have a quadratic factorization at (2,2).

In the simplest case the quadratic factor is a product of two linear factors.

Search for new integrable maps by requiring the factorization of at least one linear factor in x at the point $(2, 2)$.



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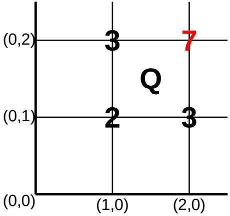


Use "corner" configuration, because computations are simpler. Also restrict to quadratic equation.

Huge algebraic problem.

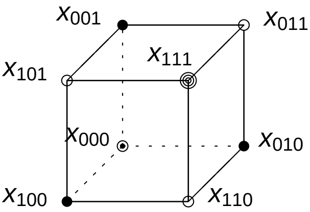
Hietarinta and Viallet, J. Phys. A: Math. Theor. **40** 12629-12643 (2007).

CAC - Consistency Around a Cube

Consistency under extensions to higher dimensions.

From 2D to 3D:

Adjoin a third direction $x_{n,m} \to x_{n,m,k}$ and construct a cube.



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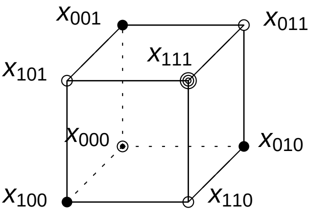
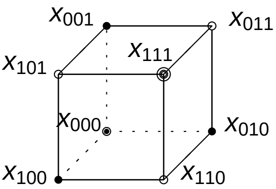


Map at the bottom $Q_{12}(x, \widetilde{x}, \widehat{x}, \widehat{\widetilde{x}}; p, q) = 0$,
on the sides $Q_{23}(x, \widehat{x}, \bar{x}, \widehat{\bar{x}}; q, r) = 0$, $Q_{31}(x, \bar{x}, \widetilde{x}, \bar{\widetilde{x}}; r, p) = 0$,
shifted maps on parallel shifted planes.

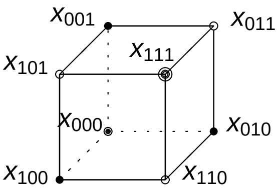


Consistency problem:

Given values at black disks, we can compute values at open disks uniquely.

But x_{111} can be computed in 3 different ways!

They must agree!



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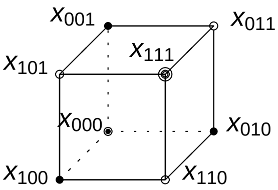
solve x_{110} from $Q_{12}(x_{000}, x_{100}, x_{010}, x_{110}; p, q) = 0,$

solve x_{011} from $Q_{23}(x_{000}, x_{010}, x_{001}, x_{011}; q, r) = 0,$

solve x_{101} from $Q_{31}(x_{000}, x_{001}, x_{100}, x_{101}; r, p) = 0,$



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\end{array}
$$

then x_{111} computed from the shifted equations

$$
\begin{array}{ll}
Q_{12}(x_{001}, x_{101}, x_{011}, x_{111}; p, q) = 0, & \text{or} \\
Q_{23}(x_{100}, x_{110}, x_{101}, x_{111}; q, r) = 0, & \text{or} \\
Q_{31}(x_{010}, x_{011}, x_{110}, x_{111}; r, p) = 0, &
\end{array}
$$

should all agree. This is consistency around the cube, CAC.

- CAC represents a rather high level of integrability.
- It is a kind of Bianchi identity [Nimmo and Schief, Proc. R. Soc. Lond. A **453** (1997) 255].
- First proposed as a property of maps in Nijhoff, Ramani, Grammaticos and Ohta, Stud. Appl. Math. **106** (2001) 261.
- It allows construction of Lax presentation [Nijhoff and Walker, Glasgow Math. J. **43A** (2001) 109].

CAC provides a Lax pair

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The third direction is taken as the spectral direction.
The auxiliary functions are generated from x_{**1}:
One solves Q_{13} for x_{101} and Q_{23} for x_{011} and the dependence on these variables is linearized by introducing f, g:
$x_{001} = f/g,\ x_{101} = f_{[1]}/g_{[1]},\ x_{011} = f_{[2]}/g_{[2]},\ \lambda = r.$

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For the discrete KdV
$(x_{n,m+1} - x_{n+1,m})(x_{n,m} - x_{n+1,m+1}) = p^2 - q^2$, we have
$Q_{13} \equiv (x_{001} - x_{100})(x_{000} - x_{101}) = p^2 - r^2$, and get

$$\frac{f_{[1]}}{g_{[1]}} = \frac{xf + (\lambda^2 - p^2 - \widetilde{x}\, x)g}{f - \widetilde{x} g},$$

$$\frac{f_{[2]}}{g_{[2]}} = \frac{xf + (\lambda^2 - q^2 - \widehat{x}\, x)g}{f - \widehat{x} g}.$$

Define $\phi = \binom{f}{g}$ and write the result

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as a matrix relation

$$\phi_{[1]} = L\phi, \quad \phi_{[2]} = M\phi$$

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For the KdV-map one finds

$$L = \gamma \begin{pmatrix} x & \lambda^2 - p^2 - x\widetilde{x} \\ 1 & -\widetilde{x} \end{pmatrix}, \quad M = \gamma' \begin{pmatrix} x & \lambda^2 - q^2 - x\widehat{x} \\ 1 & -\widehat{x} \end{pmatrix}.$$

where γ, γ' are separation constants.
The consistency condition $\phi_{[12]} = \phi_{[21]}$, i.e., $L_{[2]}M = M_{[1]}L$, determines the constants γ, γ' and also yields the map $(\widehat{x} - \widetilde{x})(x - \widehat{\widetilde{x}}) = p^2 - q^2$.

CAC as a search method

CAC has been used as a method to search and classify lattice equations:

Adler, Bobenko and Suris, Commun.Math.Phys. **233** 513 (2003)

with 2 additional assumptions:

- symmetry ($\varepsilon, \sigma = \pm 1$):

$$Q(x_{000}, x_{100}, x_{010}, x_{110}; p_1, p_2) = \varepsilon Q(x_{000}, x_{010}, x_{100}, x_{110}; p_2, p_1) = \sigma Q(x_{100}, x_{000}, x_{110}, x_{010}; p_1, p_2)$$

- "tetrahedron property": x_{111} does not depend on x_{000}.

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- “tetrahedron property”: x_{111} does not depend on x_{000}.

Result: complete classification under these assumptions, 9 models.

ABS results:

List H:

(H1) $$(x-\hat{\tilde{x}})(\tilde{x}-\hat{x})+q-p=0,$$

(H2) $$(x-\hat{\tilde{x}})(\tilde{x}-\hat{x})+(q-p)(x+\tilde{x}+\hat{x}+\hat{\tilde{x}})+q^2-p^2=0,$$

(H3) $$p(x\tilde{x}+\hat{x}\hat{\tilde{x}})-q(x\hat{x}+\tilde{x}\hat{\tilde{x}})+\delta(p^2-q^2)=0.$$

List A:

(A1) $$p(x+\hat{x})(\tilde{x}+\hat{\tilde{x}})-q(x+\tilde{x})(\hat{x}+\hat{\tilde{x}})-\delta^2pq(p-q)=0,$$

(A2)
$$(q^2-p^2)(x\tilde{x}\hat{x}\hat{\tilde{x}}+1)+q(p^2-1)(x\hat{x}+\tilde{x}\hat{\tilde{x}})-p(q^2-1)(x\tilde{x}+\hat{x}\hat{\tilde{x}})=0.$$

Main list:

(Q1) $$p(x-\hat{x})(\tilde{x}-\hat{\tilde{x}})-q(x-\tilde{x})(\hat{x}-\hat{\tilde{x}})+\delta^2 pq(p-q)=0,$$

(Q2)
$$p(x-\hat{x})(\tilde{x}-\hat{\tilde{x}})-q(x-\tilde{x})(\hat{x}-\hat{\tilde{x}})+pq(p-q)(x+\tilde{x}+\hat{x}+\hat{\tilde{x}})$$
$$-pq(p-q)(p^2-pq+q^2)=0,$$

(Q3)
$$(q^2-p^2)(x\hat{\tilde{x}}+\tilde{x}\hat{x})+q(p^2-1)(x\tilde{x}+\hat{x}\hat{\tilde{x}})-p(q^2-1)(x\hat{x}+\tilde{x}\hat{\tilde{x}})$$
$$-\delta^2(p^2-q^2)(p^2-1)(q^2-1)/(4pq)=0,$$

(Q4) (the root model from which others follow)
$$a_0 x\tilde{x}\hat{x}\hat{\tilde{x}}+a_1(x\tilde{x}\hat{x}+\tilde{x}\hat{x}\hat{\tilde{x}}+\hat{x}\hat{\tilde{x}}x+\hat{\tilde{x}}x\tilde{x})+a_2(x\hat{\tilde{x}}+\tilde{x}\hat{x})+$$
$$\bar{a}_2(x\tilde{x}+\hat{x}\hat{\tilde{x}})+\tilde{a}_2(x\hat{x}+\tilde{x}\hat{\tilde{x}})+a_3(x+\tilde{x}+\hat{x}+\hat{\tilde{x}})+a_4=0,$$

where the a_i depend on the lattice directions and are given in terms of Weierstrass elliptic functions. This was first derived by Adler as a superposition rule of BT's for the Krichever-Novikov equation. [Adler, Intl. Math. Res. Notices, # 1 (1998) 1-4]

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- $$x + x_{[1]} + x_{[2]} + x_{[12]} = 0$$
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Result: The above are linearizable, thus nothing new.

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Additional result: a simpler Jacobi form for (Q4) of ABS:

$$(h_1 f_2 - h_2 f_1)[(xx_{[1]}x_{[12]}x_{[2]} + 1)f_1 f_2 - (xx_{[12]} + x_{[1]}x_{[2]})] + (f_1^2 f_2^2 - 1)[(xx_{[1]} + x_{[12]}x_{[2]})f_1 - (xx_{[2]} + x_{[1]}x_{[12]})f_2] = 0,$$

$h_i^2 = f_i^4 + \delta f_i^2 + 1$, parametrizable by Jacobi elliptic functions.

A further result (JH, JPhysA, **37** L67 (2004))

$$\frac{x+e_2}{x+e_1}\,\frac{x_{[12]}+o_2}{x_{[12]}+o_1}=\frac{x_{[1]}+e_2}{x_{[1]}+o_1}\,\frac{x_{[2]}+o_2}{x_{[2]}+e_1}.$$

Note that the parameters and variables appear symmetrically.

This model has interesting geometric interpretation as it describes some special relation between eight points on a conic (Adler, `nlin.SI/0409065`).

Also this is linearizable.

Hirota’s bilinear method

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Hirota's bilinear form is well suited for constructing soliton solutions, because the dependent variable is then a polynomial of exponentials with linear exponents.

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The H1 equation is given by

$$\mathrm{H1}\equiv(u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})-(p-q)=0,$$

then the side-equations are

$$(\widetilde{u}-u)^2=r-p,\quad(\widehat{u}-u)^2=r-q.$$

The background solution

First problem in the perturbative approach:
What is the background solution?

Atkinson: Take the CAC cube and insist that the solution is a fixed point of the bar shift. The "side"-equations are then

$$Q(u,\widetilde{u},u,\widetilde{u};p,r)=0,\quad Q(u,\widehat{u},u,\widehat{u};q,r)=0.$$

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For convenience we reparametrize $(p,q)\to(a,b)$ by

$$p=r-a^2,\quad q=r-b^2.$$

The side-equations then factorize as

$$(\widetilde{u} - u - a)(\widetilde{u} - u + a) = 0, \quad (\widehat{u} - u - b)(\widehat{u} - u + b) = 0,$$

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From consistency $\theta, \in \{n, 0\}$,$\chi, \in \{m, 0\}$.
The set of possible background solution turns out to be

$$\begin{aligned} an + bm + \gamma, \\ \tfrac{1}{2}(-1)^n a + bm + \gamma, \\ an + \tfrac{1}{2}(-1)^m b + \gamma, \\ \tfrac{1}{2}(-1)^n a + \tfrac{1}{2}(-1)^m b + \gamma. \end{aligned}$$

1SS

The BT generating 1SS for H1 is

$$(u - \bar{\tilde{u}})(\tilde{u} - \bar{u}) = p - \varkappa,$$

$$(u - \bar{\hat{u}})(\bar{u} - \hat{u}) = \varkappa - q.$$

Here u is the background solution $an + bm + \gamma$, $\bar{u}$ is the new 1SS, and $\varkappa$ is the soliton parameter (the parameter in the bar-direction).

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Here u is the background solution $an+bm+\gamma$, $\bar{u}$ is the new 1SS, and $\varkappa$ is the soliton parameter (the parameter in the bar-direction).

We search for a new solution $\bar{u}$ of the form

$$\bar{u}=\bar{u}_0+v,$$

where $\bar{u}_0$ is the bar-shifted background solution

$$\bar{u}_0=an+bm+k+\lambda.$$

For v we then find

$$\widetilde{v} = \frac{Ev}{v+F}, \quad \widehat{v} = \frac{Gv}{v+H},$$

where

$$E = -(a+k), \quad F = -(a-k), \quad G = -(b+k), \quad H = -(b-k),$$

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Introducing $v = f/g$ and $\Phi = (g, f)^T$ we can write this as a matrix equation

$$\Phi(n+1, m) = \mathcal{N}(n, m)\Phi(n, m), \quad \Phi(n, m+1) = \mathcal{M}(n, m)\Phi(n, m),$$

where

$$\mathcal{N}(n, m) = \Lambda \begin{pmatrix} E & 0 \\ 1 & F \end{pmatrix}, \quad \mathcal{M}(n, m) = \Lambda' \begin{pmatrix} G & 0 \\ 1 & H \end{pmatrix},$$

In this case E, F, G, H are constants and we can choose $\Lambda = \Lambda' = 1$.

Since the matrices $\mathcal{N}, \mathcal{M}$ commute it is easy to find

$$\Phi(n,m) = \begin{pmatrix} E^n G^m & 0 \\ \frac{E^n G^m - F^n H^m}{-2k} & F^n H^m \end{pmatrix} \Phi(0,0).$$

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If we let

$$\rho_{n,m} = \left(\frac{E}{F}\right)^n \left(\frac{G}{H}\right)^m \rho_{0,0} = \left(\frac{a+k}{a-k}\right)^n \left(\frac{b+k}{b-k}\right)^m \rho_{0,0},$$

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then we obtain

$$v_{n,m} = \frac{-2k\,\rho_{n,m}}{1+\rho_{n,m}}.$$

Finally we obtain the 1SS for H1:

$$u_{n,m}^{(1SS)} = (an + bm + \lambda) + k + \frac{-2k\,\rho_{n,m}}{1+\rho_{n,m}}.$$

Bilinearizing transformation

In an explicit form the 1SS is

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We find

$$\begin{aligned} \mathrm{H1} &\equiv (u - \widehat{\widetilde{u}})(\widetilde{u} - \widehat{u}) - p + q \\ &= -\left[\mathcal{H}_1 + (a-b) f\widehat{\widetilde{f}}\right]\left[\mathcal{H}_2 + (a+b)\widehat{f}\widetilde{f}\right] / (f\widehat{f}\widetilde{f}\widehat{\widetilde{f}}) + (a^2 - b^2), \end{aligned}$$

where

$$\begin{aligned} \mathcal{H}_1 &\equiv \widehat{g}\widetilde{f} - \widetilde{g}\widehat{f} + (a-b)(\widehat{f}\widetilde{f} - f\widehat{\widetilde{f}}) = 0, \\ \mathcal{H}_2 &\equiv g\widehat{\widetilde{f}} - \widehat{\widetilde{g}}f + (a+b)(f\widehat{\widetilde{f}} - \widehat{f}\widetilde{f}) = 0. \end{aligned}$$

Casoratians

For given functions $\varphi_i(n, m, h)$ we define the column vectors

$$\varphi(n, m, h) = (\varphi_1(n, m, h), \varphi_2(n, m, h), \cdots, \varphi_N(n, m, h))^T,$$

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For example

$$\begin{aligned} C^1_{n,m}(\varphi) &:= |\varphi(n,m,0), \varphi(n,m,1), \cdots, \varphi(n,m,N-1)| \\ &\equiv |0, 1, \cdots, N-1| \equiv |\widehat{N-1}|, \\ C^2_{n,m}(\varphi) &:= |\varphi(n,m,0), \cdots, \varphi(n,m,N-2), \varphi(n,m,N)| \\ &\equiv |0, 1, \cdots, N-2, N| \equiv |\widehat{N-2}, N|, \end{aligned}$$

Main result

The bilinear equations $\mathcal{H}_i$ are solved by Casoratians
$f = |\widehat{N-1}|_{[h]}$, $g = |\widehat{N-2}, N|_{[h]}$, with ψ given by

$$\psi_i(n,m,l;k_i) = \varrho_i^+ k_i^h (a+k_i)^n (b+k_i)^m + \varrho_i^- (-k_i)^h (a-k_i)^n (b-k_i)^m.$$

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Similar results exist for H2,H3,Q1,Q3

J. Hietarinta and D.J. Zhang, *Soliton solutions for ABS lattice equations: II Casoratians and bilinearization*
to appear in J. Phys. A: Math. Theor. `arXiv:0903.1717`.

J. Atkinson, J. Hietarinta and F. Nijhoff, *Soliton solutions for Q3*, J. Phys. A: Math. Theor., **41** 142001 (2008).
`arXiv:0801.0806`

The structure of the soliton solution is similar to those of the Hirota-Miwa equation

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    Linear growth = linearizability
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  - Generic
- Consistency-Around-Cube
  - Applicable only to maps defined on a square lattice.
  - Strong: Lax pair follows immediately
  - Soliton solutions can be constructed systematically