Frobenius integrable decompositions for PDEs

GAO Liang

Department of Applied Mathematics,
School of Science,
Northwestern Polytechnical University
Xi'an, Shaanxi 710129, P.R. China
gaoliang_box@126.com

Joint work with Wen-Xiu Ma and Wei Xu

Outline

1. **Introduction**
2. **Specific PDEs possessing Frobenius integrable decompositions (FIDs)**
3. **Conclusions**

1. Introduction

- Many PDEs exhibiting soliton phenomena appeared in many science subjects —— ***fluid physics, solid physics, elementary particle physics, biological physics, superconductor physics, …***

- It is a quite fascinating research topic that how to solve such PDEs to obtain interesting solutions including ***solitons***, which attracts much attention of mathematicians, physicists and dynamicists.

- In the past several decades, there have been many efficient methods for constructing exact solutions to nonlinear PDEs.
- Though the solving methods are diverse, appropriate **reductions**

  —— ***similarity reductions***,

  —— ***symmetry constraints***,

  —— ***travelling wave reductions***

  …

- To reduce given PDEs to **simpler** PDEs and/or integrable ODEs.

- W.X. Ma, H.Y. Wu, J.S. He, Partial differential equations possessing Frobenius integrable decompositions, *Physics Letters A*, 364 (2007) 29-32.
- **They presented FIDs for two classes of nonlinear evolution equations (NEEs) with logarithmic derivative Bäcklund transformations in soliton theory.**

$$u = (\ln\phi)_x = \frac{\psi}{\phi}, \qquad u = (\ln\phi)_{xx} = \frac{\lambda\phi^2 - \psi^2}{\phi^2}.$$

- **The discussed NEEs are transformed into systems of Frobenius integrable ODEs with cubic nonlinearity.**

- F.C. You, T.C. Xia, J. Zhang, Frobenius integrable decompositions for two classes of nonlinear evolution equations with variable coefficients, *Modern Physics Letters B*, 23 (12) (2009) 1519-1524

- They obtained two classes of PDEs with variable coefficients possessing **FIDs**, including

  - **the KdV equation**
  - **the potential KdV equation**
  - **the Boussinesq equation**
  - **the generalized BBM equation, …**

What is FIDs ?

$$P(u, u_t, u_x, u_{xt}, \cdots) = 0$$

$$u = v(\Phi) = v(\Phi(x,t)),$$

$$\Phi_x = \mathrm{A}(\Phi), \qquad \Phi_t = \mathrm{B}(\Phi),$$

- Then we say that the **equation** possesses a FID.

- FIDs generalize compatible time-space decompositions requiring Hamiltonian structures, which aim to guarantee the Liouville integrability ([1, 2]).

- [1] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1989.
- [2] W.X. Ma, in: A. Scott (Ed.), *Encyclopedia of Nonlinear Science*, Taylor & Francis, New York, 2005, pp. 450-453.

- Through FIDs, **a PDE problem can be transformed into two associated ODE problems**. Thus, the existence of solutions can be guaranteed easily by the theory of ODEs.

Our work

- Present two classes of PDEs of specific type possessing FIDs by introducing some general ansatzes on FIDs, motivated by the works about FIDs by Ma and You et al.
- Two kinds of functions for Bäcklund transformations are taken in our constructive computation algorithm, and the associated Frobenius integrable ODEs possess **higher-order nonlinearity**.

2. Specific PDEs possessing FIDs

$$R(u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{5x}, u_{7x}, u_{9x}, u_t, u_{tt}, u_{xxt}, \cdots) = 0.$$

Many interesting wave equations belong to this class of PDEs,

- ✧ *the KdV,*
- ✧ *the potential KdV,*
- ✧ *the generalized seventh-order KdV,*
- ✧ *the Burgers,*
- ✧ *the spatially periodic third-order dispersive PDE,*
- ✧ *the b-equation,*

…

Based on the **symmetry constrains theory** [1-4],

- [1] W.X. Ma, W. Strampp, *Physics Letters A*, 185 (1994) 277.
- [2] W.X. Ma, *Journal of the Physical Society of Japan*, 64 (1995) 1085.
- [3] W.X. Ma, Z.X. Zhou, *Journal of Mathematical Physics*, 42 (2001) 4345.
- [4] Y.B. Zeng, W.X. Ma, *Journal of Mathematical Physics*, 40 (1999) 6526.

we consider **the following case**

$$\Phi=(\phi,\psi)^T=(\phi(x,t),\psi(x,t))^T,$$

- which satisfies **two Frobenius integrable systems**:

a real parameter

$$\phi_x=\psi,\qquad \psi_x=\lambda\phi,$$

undetermined polynomials

$$\phi_t=\theta_1(\phi,\psi),\qquad \psi_t=\theta_2(\phi,\psi).$$

It is easy to see that the **equation** can be generated from the Schrödinger spectral problem with zero potential.

- Then, we have the following mixed derivatives for the considered FIDs:

$$\begin{cases} \phi_{xt} = \psi_t = \theta_2(\phi, \psi), \\ \phi_{tx} = \theta_{1,\phi}\psi + \lambda\theta_{1,\psi}\phi, \end{cases}$$

and

$$\begin{cases} \psi_{xt} = \lambda\phi_t = \lambda\theta_1(\phi, \psi), \\ \psi_{tx} = \theta_{2,\phi}\psi + \lambda\theta_{2,\psi}\phi. \end{cases}$$

- Thus, we get

$$\theta_2(\phi,\psi) = \theta_{1,\phi}\psi + \lambda\theta_{1,\psi}\phi,$$

and accordingly,

$$\lambda\theta_1(\phi,\psi) = \theta_{1,\phi\phi}\psi^2 + 2\lambda\theta_{1,\phi\psi}\phi\psi + \lambda^2\theta_{1,\psi\psi}\phi^2 + \lambda(\theta_{1,\phi}\phi + \theta_{1,\psi}\psi).$$

↓

the only condition on θ_1

To search for θ_1 which satisfies

$$\lambda\theta_1(\phi,\psi) = \theta_{1,\phi\phi}\psi^2 + 2\lambda\theta_{1,\phi\psi}\phi\psi + \lambda^2\theta_{1,\psi\psi}\phi^2 + \lambda(\theta_{1,\phi}\phi + \theta_{1,\psi}\psi).$$

Take the following ansatze

natural numbers

$$\theta_1 = \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} b_{i,j}\phi^i\psi^j,$$

arbitrary constants

when $m_1 = m_2 = 7$

$$\begin{aligned}
\theta_1 = & -\lambda^3 b_{1,6}\phi^7 - \lambda^3 b_{0,7}\phi^6\psi + 3\lambda^2 b_{1,6}\phi^5\psi^2 + 3\lambda^2 b_{0,7}\phi^4\psi^3 - 3\lambda b_{1,6}\phi^3\psi^4 - 3\lambda b_{0,7}\phi^2\psi^5 \\
& + b_{1,6}\phi\psi^6 + b_{0,7}\psi^7 + \lambda^2 b_{1,4}\phi^5 + \lambda^2 b_{0,5}\phi^4\psi - 2\lambda b_{1,4}\phi^3\psi^2 - 2\lambda b_{0,5}\phi^2\psi^3 + b_{1,4}\phi\psi^4 \\
& + b_{0,5}\psi^5 - \lambda b_{1,2}\phi^3 - \lambda b_{0,3}\phi^2\psi + b_{1,2}\phi\psi^2 + b_{0,3}\psi^3 + b_{1,0}\phi + b_{0,1}\psi.
\end{aligned}$$

$$\begin{aligned}
\theta_2 = & \lambda(-\lambda^3 b_{0,7}\phi^6 + 6\lambda^2 b_{1,6}\phi^5\psi + 9\lambda^2 b_{0,7}\phi^4\psi^2 - 12\lambda b_{1,6}\phi^3\psi^3 - 15\lambda b_{0,7}\phi^2\psi^4 + 6b_{1,6}\phi\psi^5 \\
& + 7b_{0,7}\psi^6 + \lambda^2 b_{0,5}\phi^4 - 4\lambda b_{1,4}\phi^3\psi - 6\lambda b_{0,5}\phi^2\psi^2 + 4b_{1,4}\phi\psi^3 + 5b_{0,5}\psi^4 - \lambda b_{0,3}\phi^2 \\
& + 2b_{1,2}\phi\psi + 3b_{0,3}\psi^2 + b_{0,1})\phi + (-7\lambda^3 b_{1,6}\phi^6 - 6\lambda^3 b_{0,7}\phi^5\psi + 15\lambda^2 b_{1,6}\phi^4\psi^2 \\
& + 12\lambda^2 b_{0,7}\phi^3\psi^3 - 9\lambda b_{1,6}\phi^2\psi^4 - 6\lambda b_{0,7}\phi\psi^5 + b_{1,6}\psi^6 + 5\lambda^2 b_{1,4}\phi^4 \\
& + 4\lambda^2 b_{0,5}\phi^3\psi - 6\lambda b_{1,4}\phi^2\psi^2 - 4\lambda b_{0,5}\phi\psi^3 + b_{1,4}\psi^4 \\
& - 3\lambda b_{1,2}\phi^2 - 2\lambda b_{0,3}\phi\psi + b_{1,2}\psi^2 + b_{1,0})\psi.
\end{aligned}$$

- On the other hand, we consider the following specific type of the polynomial R

$$
\begin{aligned}
R(u,u_x,u_{xx},u_{xxx},u_{xxxx},u_{5x},u_{7x},u_{9x},u_t,u_{tt},u_{xxt},\cdots) \\
= d_{0,0}u + d_{0,1}u_x + d_{0,2}u_{xx} + d_{0,3}u_{xxx} + d_{0,4}u_{xxxx} + d_{1,0}u^2 + d_{1,1}uu_x + d_{1,2}uu_{xx} + d_{1,3}uu_{xxx} \\
+ d_{1,4}uu_{xxxx} + d_{2,0}u_x^2 + d_{2,1}u_xu_{xx} + d_{2,2}u_xu_{xxx} + d_{2,3}u_xu_{xxxx} + d_{3,0}u_{xx}^2 + d_{3,1}u_{xx}u_{xxx} \\
+ d_{3,2}u_{xx}u_{xxxx} + d_{4,0}u_{xxx}^2 + d_{4,1}u_{xxx}u_{xxxx} + d_{5,0}u_{xxxx}^2 + e_0u_{5x} + e_1u_{7x} + e_2u_{9x} \\
+ e_3uu_{5x} + e_4uu_{7x} + e_5uu_{9x} + f_0u^2u_x + f_1u^2u_{xxx} + f_2u^2u_{5x} + f_3u_x^3 \\
+ f_4u^3u_x + f_5uu_xu_{xx} + g_0u_t + g_1u_{tt} + g_2u_{xxt} + g_3u_{xxt}^2.
\end{aligned}
$$

- **Theorem (a)** If we take a Bäcklund transformation

$$u = (\ln\phi)_x = \frac{\psi}{\phi}$$

from the Frobenius integrable systems

$$\phi_x = \psi, \qquad \psi_x = \lambda\phi,$$

$$\phi_t = \theta_1(\phi,\psi), \qquad \psi_t = \theta_2(\phi,\psi),$$

to the PDE

$$P(u, u_t, u_x, u_{xt}, \cdots) = 0,$$

$$\theta_1 = -\lambda^3 b_{1,6}\phi^7 + 3\lambda^2 b_{1,6}\phi^5\psi^2 - 3\lambda b_{1,6}\phi^3\psi^4 + b_{1,6}\phi\psi^6 + \lambda^2 b_{1,4}\phi^5 - 2\lambda b_{1,4}\phi^3\psi^2$$
$$+ b_{1,4}\phi\psi^4 - \lambda b_{1,2}\phi^3 + b_{1,2}\phi\psi^2 + b_{1,0}\phi + b_{0,1}\psi\,,$$

$$\theta_2 = \lambda(6\lambda^2 b_{1,6}\phi^5\psi - 12\lambda b_{1,6}\phi^3\psi^3 + 6b_{1,6}\phi\psi^5 - 4\lambda b_{1,4}\phi^3\psi + 4b_{1,4}\phi\psi^3 + 2b_{1,2}\phi\psi + b_{0,1})\phi$$
$$+ (15\lambda^2 b_{1,6}\phi^4\psi^2 - 7\lambda^3 b_{1,6}\phi^6 - 9\lambda b_{1,6}\phi^2\psi^4 + b_{1,6}\psi^6 + 5\lambda^2 b_{1,4}\phi^4 - 6\lambda b_{1,4}\phi^2\psi^2$$
$$+ b_{1,4}\psi^4 - 3\lambda b_{1,2}\phi^2 + b_{1,2}\psi^2 + b_{1,0})\psi\,,$$

$$
\begin{aligned}
R = {} & (224\lambda^2 e_4 + 8\lambda d_{2,3} - 40\lambda e_3 + 24 d_{0,4} - 2d_{2,1} - 6d_{1,3})u^3 u_x + f_1 u^2 u_{xxx} + f_2 u^2 u_{5x} + f_3 u_x^3 \\
& + d_{1,4} u u_{xxxx} + d_{1,3} u u_{xxx} + (3d_{2,2} - 40\lambda f_2 - 60 e_0 + 2d_{3,0} - 3f_1 - 1/2 f_3 + 12 d_{1,4}) u u_x u_{xx} \\
& + d_{1,2} u u_{xx} + e_3 u u_{5x} + e_4 u u_{7x} + 630 e_2 u_{xxxx}^2 - 35 e_4 u_{xxx} u_{xxxx} + d_{2,3} u_x u_{xxxx} + d_{4,0} u_{xxx}^2 \\
& + d_{2,2} u_x u_{xxx} + d_{3,0} u_{xx}^2 + d_{2,1} u_x u_{xx} - (256\lambda^4 e_2 + 64\lambda^3 e_1 + 16\lambda^3 f_2 + 16\lambda^2 e_0 \\
& + 4\lambda^2 f_1 - 8\lambda^2 d_{1,4} + 4\lambda b_{0,1} g_2 + 4\lambda d_{0,3} - 2\lambda d_{1,2} + b_{0,1} g_0 + d_{0,1}) u^2 u_x / \lambda \\
& + (24\lambda^2 e_3 - 288\lambda^3 e_4 - 8\lambda^2 d_{2,3} + 2b_{0,1}^2 g_1 + 2\lambda d_{2,1} - 16\lambda d_{0,4} + 2\lambda d_{1,3} \\
& + 2d_{0,2}) u u_x + (105 e_1 - 3/4 d_{4,0} + 5/2 f_2 - 3/4 b_{0,1}^2 g_3) u_{xx} u_{xxxx} \\
& + d_{0,4} u_{xxxx} - (140\lambda e_4 + 2d_{2,3} + 10 e_3) u_{xx} u_{xxx} + d_{0,3} u_{xxx} \\
& + d_{0,2} u_{xx} + e_2 u_{9x} + e_1 u_{7x} + e_0 u_{5x} + d_{0,1} u_x - (4\lambda^3 c_{0,1}^2 g_3 \\
& + 7936\lambda^4 e_2 + 4\lambda^3 d_{4,0} - 272\lambda^3 e_1 + 16\lambda^2 e_0 \\
& - 2\lambda^2 d_{2,2} + \lambda^2 f_3 - 2\lambda b_{0,1} g_2 - 2\lambda d_{0,3} \\
& + b_{0,1} g_0 + d_{0,1}) u_x^2 / \lambda + g_0 u_t + g_1 u_{tt} \\
& + g_2 u_{xxt} + g_3 u_{xxt}^2 .
\end{aligned}
$$

- **(b)** If we take a Bäcklund transformation

$$u = (\ln\phi)_{xx} = \frac{\lambda\phi^2 - \psi^2}{\phi^2}$$

$$\theta_1 = -\lambda^3 b_{1,6}\phi^7 + 3\lambda^2 b_{1,6}\phi^5\psi^2 - 3\lambda b_{1,6}\phi^3\psi^4 + b_{1,6}\phi\psi^6 + \lambda^2 b_{1,4}\phi^5 - 2\lambda b_{1,4}\phi^3\psi^2$$
$$+ b_{1,4}\phi\psi^4 - \lambda b_{1,2}\phi^3 + b_{1,2}\phi\psi^2 + b_{1,0}\phi + b_{0,1}\psi,$$

$$\theta_2 = -\lambda^3 b_{1,6}\phi^6\psi + 3\lambda^2 b_{1,6}\phi^4\psi^3 - 3\lambda b_{1,6}\phi^2\psi^5 + \lambda^2 b_{1,4}\phi^4\psi - 2\lambda b_{1,4}\phi^2\psi^3$$
$$- \lambda b_{1,2}\phi^2\psi + \lambda b_{0,1}\phi + b_{1,6}\psi^7 + b_{1,4}\psi^5 + b_{1,2}\psi^3 + b_{0,1}\psi,$$

$$
\begin{aligned}
&R = \\
&f_4u^3u_x + f_1u^2u_{xxx} + f_5uu_xu_{xx} + (443520\lambda^2e_2 - 352\lambda^2d_{4,1} - 3584\lambda^2e_4 - 120\lambda e_3 + 8\lambda f_1 - \lambda f_4 + 2\lambda f_5 \\
&-360e_0 + 12d_{1,3} + 6d_{2,1})u^2u_x + (4d_{4,1} - 5040e_2 + 56e_4)u^2u_{5x} + e_4uu_{7x} + e_3uu_{5x} + d_{1,4}uu_{xxxx} + d_{1,3}uu_{xxx} \\
&+d_{1,2}uu_{xx} + (240\lambda d_{4,1} - 302400\lambda e_2 + 1680\lambda e_4 + 30d_{2,3} + 18d_{3,1} - 5040e_1 + 90e_3 - 3f_1 + 1/4f_4 \\
&-3/2f_5)u_x^3 + d_{2,3}u_xu_{xxxx} + d_{2,1}u_xu_{xx} + d_{3,2}u_{xx}u_{xxxx} + d_{4,1}u_{xxx}u_{xxxx} + d_{3,0}u_{xx}^2 + d_{3,1}u_{xx}u_{xxx} + (32\lambda^3d_{3,2} \\
&-16\lambda^2d_{1,4} - 4\lambda^2d_{3,0} + 6b_{0,1}^2g_1 + 2\lambda d_{1,2} + 6d_{0,2})u^2 + (65280\lambda^3e_2 - 64\lambda^3d_{4,1} - 64\lambda^3e_4 \\
&-16\lambda^2d_{2,3} - 16\lambda^2d_{3,1} + 4032\lambda^2e_1 - 16\lambda^2e_3 + 240\lambda e_0 - 4\lambda d_{1,3} - 4\lambda d_{2,1} + 12b_{0,1}g_2 \\
&+12d_{0,3})uu_x + (10\lambda d_{1,4} - 4\lambda^2d_{3,2} + 30d_{0,4} - 3/2d_{1,2})u_x^2 - (5/2d_{1,4} + 3/4d_{3,0}) \\
&u_xu_{xxx} + e_2u_{9x} + e_1u_{7x} + e_0u_{5x} + d_{0,4}u_{xxxx} + d_{0,3}u_{xxx} + d_{0,2}u_{xx} - (b_{0,1}^2g_3 + 5/4d_{3,2}) \\
&u_{xxx}^2 - (4\lambda b_{0,1}^2g_1 + 16\lambda^2d_{0,4} + 4\lambda d_{0,2})u - (256\lambda^4e_2 + 64\lambda^3e_1 + 16\lambda^2e_0 \\
&+4\lambda b_{0,1}g_2 + 4\lambda d_{0,3} + b_{0,1}g_0)u_x + g_0u_t + g_1u_{tt} + g_2u_{xxt} + g_3u_{xxt}^2.
\end{aligned}
$$

- Take **a special reduction** as follows:

$$
\begin{aligned}
&R(u,u_x,u_{xx},u_{xxx},u_{xxxx},u_{5x},u_{7x},u_{9x},u_t,u_{tt},u_{xxt},\cdots)\\
&= d_{0,0}u + d_{0,1}u_x + d_{0,2}u_{xx} + d_{0,3}u_{xxx} + d_{0,4}u_{xxxx} + d_{1,1}uu_x + d_{1,2}uu_{xx} + d_{1,3}uu_{xxx}\\
&\quad + d_{2,0}u_x^2 + d_{2,1}u_xu_{xx} + d_{2,2}u_xu_{xxx} + d_{2,3}u_xu_{xxxx} + d_{3,1}u_{xx}u_{xxx} + e_0u_{5x} + e_1u_{7x}\\
&\quad + e_2u_{9x} + e_3uu_{5x} + f_0u^2u_x + f_1u^2u_{xxx} + f_2u^2u_{5x} + f_3u_x^3 + f_4u^3u_x\\
&\quad + f_5uu_xu_{xx} + g_0u_t + g_1u_{tt} + g_2u_{xxt}.
\end{aligned}
$$

For the $u = (\ln\phi)_x = \dfrac{\psi}{\phi}$

The corresponding results

$$\theta_1 = -\lambda^3 b_{1,6}\phi^7 + 3\lambda^2 b_{1,6}\phi^5\psi^2 - 3\lambda b_{1,6}\phi^3\psi^4 + b_{1,6}\phi\psi^6 + \lambda^2 b_{1,4}\phi^5 - 2\lambda b_{1,4}\phi^3\psi^2 + b_{1,4}\phi\psi^4 - \lambda b_{1,2}\phi^3 + b_{1,2}\phi\psi^2 + b_{1,0}\phi + b_{0,1}\psi,$$

$$\theta_2 = \lambda(6\lambda^2 b_{1,6}\phi^5\psi - 12\lambda b_{1,6}\phi^3\psi^3 + 6b_{1,6}\phi\psi^5 - 4\lambda b_{1,4}\phi^3\psi + 4b_{1,4}\phi\psi^3 + 2b_{1,2}\phi\psi + b_{0,1})\phi + (-7\lambda^3 b_{1,6}\phi^6 + 15\lambda^2 b_{1,6}\phi^4\psi^2 - 9\lambda b_{1,6}\phi^2\psi^4 + b_{1,6}\psi^6 + 5\lambda^2 b_{1,4}\phi^4 - 6\lambda b_{1,4}\phi^2\psi^2 + b_{1,4}\psi^4 - 3\lambda b_{1,2}\phi^2 + b_{1,2}\psi^2 + b_{1,0})\psi,$$

$$
\begin{aligned}
R = {} & (8\lambda d_{2,3} - 40\lambda e_3 + 24 d_{0,4} - 6 d_{1,3} - 2 d_{2,1}) u^3 u_x + f_1 u^2 u_{xxx} - 42 e_1 u^2 u_{5x} + f_3 u_x^3 \\
& + (336\lambda^2 e_1 - 2\lambda d_{2,2} - 4\lambda f_1 + \lambda f_3 - 6 b_{0,1} g_2 - 6 d_{0,3} + 2 d_{1,2} + d_{2,0}) u^2 u_x \\
& + d_{1,2} u u_{xx} + (1680\lambda e_1 + 3 d_{2,2} - 60 e_0 - 3 f_1 - 1/2\, f_3) u u_x u_{xx} + d_{1,3} u u_{xxx} \\
& + e_3 u u_{5x} + d_{2,1} u_x u_{xx} + d_{2,2} u_x u_{xxx} + d_{2,3} u_x u_{xxxx} + d_{2,0} u_x^2 + (24\lambda^2 e_3 \\
& - 8\lambda^2 d_{2,3} + 2 b_{0,1}{}^2 g_1 - 16\lambda d_{0,4} + 2\lambda d_{1,3} + 2\lambda d_{2,1} + 2 d_{0,2}) u u_x \\
& - (2 d_{2,3} + 10 e_3) u_{xx} u_{xxx} + d_{0,2} u_{xx} + d_{0,3} u_{xxx} + d_{0,4} u_{xxxx} \\
& + e_0 u_{5x} + e_1 u_{7x} + (272\lambda^3 e_1 + 2\lambda^2 d_{2,2} - 16\lambda^2 e_0 \\
& - \lambda^2 f_3 + 2\lambda b_{0,1} g_2 + 2\lambda d_{0,3} - \lambda d_{2,0} \\
& - b_{0,1} g_0) u_x + g_0 u_t + g_1 u_{tt} \\
& + g_2 u_{xxt}.
\end{aligned}
$$

- It includes many nonlinear equations,
- **the potential KdV equation**

$$g_0 u_t + d_{0,3} u_{xxx} + 6 d_{0,3} u_x^2 = 0,$$

- **the Burgers equation**

$$g_0 u_t + 2 d_{0,2} u u_x + d_{0,2} u_{xx} = 0,$$

- **b - equation**

$$g_0 u_t - \frac{1}{2\lambda} u_{xxt} - 2 u u_x = \frac{1}{2\lambda}(-3 u_x u_{xx} + u u_{xxx}).$$

a general case of the famous Rod equation [1]

[1] H.H. Dai, Y. Huo, *Proceedings of the Royal Society of London Series A*, 456 (2000) 331.

For the $u = (\ln\phi)_{xx} = \dfrac{\lambda\phi^2 - \psi^2}{\phi^2}$

The corresponding results

$$\theta_1 = -\lambda^3 b_{1,6}\phi^7 + 3\lambda^2 b_{1,6}\phi^5\psi^2 - 3\lambda b_{1,6}\phi^3\psi^4 + b_{1,6}\phi\psi^6 + \lambda^2 b_{1,4}\phi^5 - 2\lambda b_{1,4}\phi^3\psi^2 + b_{1,4}\phi\psi^4 - \lambda b_{1,2}\phi^3 + b_{1,2}\phi\psi^2 + b_{1,0}\phi + b_{0,1}\psi,$$

$$\theta_2 = -\lambda^3 b_{1,6}\phi^6\psi + 3\lambda^2 b_{1,6}\phi^4\psi^3 - 3\lambda b_{1,6}\phi^2\psi^5 + \lambda^2 b_{1,4}\phi^4\psi - 2\lambda b_{1,4}\phi^2\psi^3 - \lambda b_{1,2}\phi^2\psi + \lambda b_{0,1}\phi + b_{1,6}\psi^7 + b_{1,4}\psi^5 + b_{1,2}\psi^3 + \psi b_{1,0},$$

$$
\begin{aligned}
R = {} & f_4 u^3 u_x + f_1 u^2 u_{xxx} + (443520\lambda^2 e_2 - 120\lambda e_3 + 8\lambda f_1 - \lambda f_4 + 2\lambda f_5 + 12 d_{1,3} + 6 d_{2,1} \\
& - 360 e_0) u^2 u_x + d_{1,2} u u_{xx} + d_{1,3} u u_{xxx} + e_3 u u_{5x} + d_{3,0} u_{xx} u_{xxx} + d_{2,1} u_x u_{xx} + (30 d_{2,3} \\
& + 18 d_{3,0} - 302400\lambda e_2 - 5040 e_1 + 90 e_3 - 3 f_1 + 1/4 f_4 - 3/2 f_5) u_x^3 + e_2 u_{9x} \\
& + d_{0,3} u_{xxx} + f_5 u u_x u_{xx} + d_{0,4} u_{xxxx} + e_0 u_{5x} + e_1 u_{7x} + (30 d_{0,4} - 3/2 d_{1,2}) u_x^2 \\
& + (4/3\lambda^2 d_{1,2} - 16\lambda^2 d_{0,4}) u - (b_{0,1}{}^2 g_1 + 1/3\lambda d_{1,2}) u_{xx} - (64\lambda^3 e_1 \\
& + 256\lambda^4 e_2 + 16\lambda^2 e_0 + 4\lambda c_{0,1} g_2 + 4\lambda d_{0,3} + b_{0,1} g_0) u_x \\
& + d_{2,3} u_x u_{xxxx} - 5040 e_2 u^2 u_{5x} + (65280\lambda^3 e_2 \\
& - 16\lambda^2 d_{2,3} - 16\lambda^2 d_{3,0} + 4032\lambda^2 e_1 \\
& - 16\lambda^2 e_3 - 4\lambda d_{1,3} - 4\lambda d_{2,1} \\
& + 240\lambda e_0 + 12 b_{0,1} g_2 \\
& + 12 d_{0,3}) u u_x + g_0 u_t \\
& + g_1 u_{tt} + g_2 u_{xxt}.
\end{aligned}
$$

It includes:

- **the KdV equation**

$$g_0 u_t + 12 d_{0,3} u u_x + d_{0,3} u_{xxx} = 0,$$

- **the family of spatially periodic third-order dispersive PDE**

$$b_{0,1} g_0 u_t - b_{0,1}^2 g_0 u_x + b_{0,1} d_{0,3} u_{xxx} - d_{0,3} u_{xxt} = b_{0,1} \alpha (\lambda u^2 + u_x^2 - \frac{1}{2} u u_{xx})_x,$$

- **the generalized seventh-order KdV equation**

$$g_0 u_t + (2\lambda f_5 - 120\lambda e_3 + 8\lambda f_1) u^2 u_x + (30 d_{2,3} + 18 d_{3,0} - \frac{3}{2} f_5 + 90 e_3 - 5040 e_1 - 3 f_1) u_x^3$$
$$+ f_5 u u_x u_{xx} + f_1 u^2 u_{xxx} + d_{3,0} u_{xx} u_{xxx} + d_{2,3} u_x u_{xxxx} + e_3 u u_{5x} + e_1 u_{7x} = 0,$$

which has two well-known special cases [1]:

⟶ **the Lax seventh-order equation**

⟶ **the Sawada-Kotera-Ito seventh-order equation**

[1] M. Ito, *Journal of the Physical Society of Japan*, 49 (1980) 771

3. Conclusions

- Through two Bäcklund transformations of dependent variables, ninth-order PDEs possessing the FIDs have been obtained.

- Two special classes of such nonlinear PDEs possessing special FIDs have been presented under a reduction.

- The obtained PDEs contain various significant nonlinear wave equations.

- The approach adopted here can be easily applied to nonlinear variable coefficient PDEs, which are quite intriguing in ***ocean dynamics, fluid mechanics, plasma physics, etc.***

- ***A question ?*** ***If we take the Möbius transformation***

$$u = (a\varphi + b\psi)/(c\varphi + d\psi), \qquad (ad - bc \neq 0)$$

instead of the logarithmic derivative type Bäcklund transformations, what we will get?

- *Thank Prof. Wen-Xiu Ma, Prof. Xingbiao Hu and Prof. Qingping Liu!*

- **Thank you for your attention!**