ON RANDOM SAMPLING IN UNIFORM HYPERGRAPHS

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Abstract. A k-graph $\mathcal{G}^{(k)}$ on vertex set $[n] = \{1, \ldots, n\}$ is said to be (ρ, ζ)-uniform if every $S \subseteq [n]$ of size $s = |S| > \zeta n$ spans $(\rho \pm \zeta)\binom{s}{k}$ edges. A 'grabbing lemma' of Mubayi and Rödl shows that this property is typically inherited locally: if $\mathcal{G}^{(k)}$ is (ρ, ζ)-uniform, then all but $\exp\{-s^{1/k}/20\}\binom{n}{s}$ sets $S \in \binom{[n]}{s}$ span (ρ, ζ')-uniform subhypergraphs $\mathcal{G}^{(k)}[S]$, where $\zeta' \to 0$ as $\zeta \to 0$, $s \geq s_0(\zeta')$ and n is sufficiently large. In this paper, we establish a grabbing lemma for a different concept of hypergraph uniformity, and infer the result above as a corollary. In particular, we improve, in the context above, the error $\exp\{-s^{1/k}/20\}$ to $\exp\{-cs\}$, for a constant $c = c(k, \zeta') > 0$.

1. Introduction

A k-graph $\mathcal{G}^{(k)} \subseteq \binom{[n]}{k}$ with vertex set $[n]$ is (ρ, ζ)-*uniform* if every $S \subseteq [n]$, $s = |S| > \zeta n$, spans $(\rho \pm \zeta)\binom{s}{k}$ edges. (Here, $\rho \pm \zeta$ denotes a quantity between $\rho - \zeta$ and $\rho + \zeta$.) It follows by definition that the induced subhypergraph $\mathcal{G}^{(k)}[S] = \mathcal{G}^{(k)} \cap \binom{S}{k}$ inherits $(\rho, \zeta/\beta)$-uniformity whenever $s \geq \beta n$. A similar inheritance 'typically' holds when $s = o(n)$, by the following result of D. Mubayi and V. Rödl [9] (which we call the 'grabbing lemma').

Theorem 1.1 (Mubayi, Rödl [9]). *For all integers k, $0 < \rho < 1$ and $\zeta' > 0$, there exist $\zeta > 0$ and integers s and n_0 so that, whenever $\mathcal{G}^{(k)}$ is a (ρ, ζ)-uniform k-graph on vertex set $[n]$, $n > n_0$, then all but* $\exp\{-s^{1/k}/20\}\binom{n}{s}$ *sets $S \in \binom{n}{s}$ span (ρ, ζ')-uniform subhypergraphs $\mathcal{G}^{(k)}[S]$.*

The first result in the direction of Theorem 1.1 is due to R. Duke and Rödl [4], who proved a similar statement for $k = 2$. They used their result to show that, if a graph G on n vertices cannot be made k-colorable by deleting $o(n^2)$ edges, then G contains a subgraph on $O(1)$ vertices which is not k-colorable, confirming a conjecture of Erdős. Theorem 1.1 extended a result of N. Alon, W. Fernandez de la Vega, R. Kannan and M. Karpinski [1], where (in the context above) $\exp\{-s^{1/k}/20\}$ is replaced by $1/40$. Here, we consider a statement (Theorem 1.5, below) like Theorem 1.1 for a different context of hypergraph 'uniformity', and will then infer Theorem 1.1 as a corollary. Before we state our main result (which requires some preparation), we make a few general remarks.

All results in this paper concern 'partite' k-graphs, where a k-graph $\mathcal{G}^{(k)}$ is ℓ-*partite* with ℓ-*partition* $V(\mathcal{G}^{(k)}) = U_1 \cup \cdots \cup U_\ell$, if each of its edges meets each U_i, $1 \leq i \leq \ell$, at most once, i.e., all edges of $\mathcal{G}^{(k)}$ *cross* the vertex partition. Theorem 1.1 is equivalent[1] to a k-partite version thereof, which we now present. For $\mathcal{G}^{(k)}$ with k-partition $U_1 \cup \cdots \cup U_k$ and for $\boldsymbol{S} = (S_1, \ldots, S_k)$, where $\emptyset \neq S_i \subseteq U_i$, $1 \leq i \leq k$, we write $\mathcal{G}^{(k)}[\boldsymbol{S}] = \mathcal{G}^{(k)} \cap \binom{S_1 \cup \cdots \cup S_k}{k}$ for the subhypergraph of $\mathcal{G}^{(k)}$ induced by $\boldsymbol{S}$, and $d_{\mathcal{G}^{(k)}}(\boldsymbol{S}) = |\mathcal{G}^{(k)}[\boldsymbol{S}]|/(|S_1| \cdots |S_k|)$ for the *density* of $\mathcal{G}^{(k)}$ w.r.t. $\boldsymbol{S}$. To conserve terminology, we say that $\mathcal{G}^{(k)}$ is (ρ, ζ)-*uniform* if for all such $\boldsymbol{S} = (S_1, \ldots, S_k)$ where $|S_i| > \zeta |U_i|$, $1 \leq i \leq k$, we have $d_{\mathcal{G}^{(k)}}(\boldsymbol{S}) = \rho \pm \zeta$. The following version of Theorem 1.1 mirrors one in [9].

Theorem 1.2. *For all integers $k \geq 2$ and $\zeta' > 0$, there exist $\zeta, c > 0$ and integers s_0 and n_0 so that, whenever $\mathcal{G}^{(k)}$ is a (ρ, ζ)-uniform k-partite k-graph with k-partition $V(\mathcal{G}^{(k)}) = U_1 \cup \cdots \cup U_k$, where $\rho \in [0, 1]$ and $|U_i| = n_i > n_0$, $1 \leq i \leq k$, then for all $s_0 \leq s_i \leq n_i$, $1 \leq i \leq k$, all but*

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[1] The equivalence follows by a standard application of the 'weak' hypergraph regularity lemma (see [9]).

$\exp\{-c\min\{s_i : 1 \le i \le k\}\}\prod_{1\le i\le k}\binom{n_i}{s_i}$ *k-tuples* $\boldsymbol{S} = (S_1,\dots,S_k)$ *of sets* $S_i \in \binom{U_i}{s_i}$, $1 \le i \le k$, *yield* (ρ,ζ')*-uniform k-partite k-graphs* $\mathcal{G}^{(k)}[\boldsymbol{S}]$.

It suffices to prove Theorem 1.2 in the case that $s_2 = n_2,\dots,s_k = n_k$. That is to say, iterating the following statement yields Theorem 1.2.

Theorem 1.3. *For all integers* $k \ge 2$ *and* $\zeta_0 > 0$*, there exist* $\zeta = \zeta_{\text{Thm. 1.3}} > 0$ *and* $c = c_{\text{Thm. 1.3}} > 0$ *and integers* $s_0 = s_{\text{Thm 1.3}}$ *and* $n_0 = n_{\text{Thm. 1.3}}$ *so that, whenever* $\mathcal{G}^{(k)}$ *is a* (ρ,ζ)*-uniform k-partite k-graph with k-partition* $V(\mathcal{G}^{(k)}) = U_1 \cup \dots \cup U_k$*, where* $\rho \in [0,1]$ *and* $|U_i| = n_i > n_0$, $1 \le i \le k$*, then for all* $s_0 \le s \le n_1$*, all but* $\exp\{-c\,s\}\binom{n_1}{s}$ *sets* $S \in \binom{U_1}{s}$ *yield* $\boldsymbol{S} = (S,U_2,\dots,U_k)$ *for which* $\mathcal{G}^{(k)}[\boldsymbol{S}]$ *is* (ρ,ζ_0)*-uniform.*

We shall deduce Theorem 1.3 (in Section 4) from our main result (Theorem 1.5, below) together with an application of a hypergraph regularity lemma of Rödl and Schacht [11] (presented in Section 3). We now prepare to state our main result.

For positive integers $j \le \ell$ and a vertex partition $V_1 \cup \dots \cup V_\ell$, an (ℓ,j)*-cylinder* $\mathcal{H}^{(j)}$ is an ℓ-partite j-uniform hypergraph with the vertex partition above, i.e., $\mathcal{H}^{(j)}$ is a subset of $K^{(j)}(V_1,\dots,V_\ell)$, the complete ℓ-partite j-graph. For a positive integer $i \le j$, let $\mathcal{K}_i(\mathcal{H}^{(j)})$ denote the family of all crossing i-element subsets which span complete subhypergraphs in $\mathcal{H}^{(j)}$. We say that an $(\ell,j-1)$-cylinder $\mathcal{H}^{(j-1)}$ *underlies* an (ℓ,j)-cylinder $\mathcal{H}^{(j)}$ if $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$. For an integer $h \le \ell$, an (ℓ,h)*-complex* $\boldsymbol{\mathcal{H}}$ is a collection of (ℓ,j)-cylinders $\{\mathcal{H}^{(j)}\}_{j=1}^h$ where $\mathcal{H}^{(1)} = V_1 \cup \dots \cup V_\ell$ and where $\mathcal{H}^{(j-1)}$ underlies $\mathcal{H}^{(j)}$ for $2 \le j \le h$. The following definition provides central density and regularity concepts of this paper.

Definition 1.4. *Let constants* $d, d_2,\dots,d_h \in [0,1]$ *and* $\varepsilon > 0$ *be given.*

1. *For a* (j,j)*-cylinder* $\mathcal{H}^{(j)}$ *with an underlying* $(j,j-1)$*-cylinder* $\mathcal{H}^{(j-1)}$*, let* $\mathcal{Q}^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$. *The* density *of* $\mathcal{H}^{(j)}$ w.r.t. $\mathcal{Q}^{(j-1)}$ *is* $d(\mathcal{H}^{(j)}|\mathcal{Q}^{(j-1)}) = |\mathcal{H}^{(j)} \cap \mathcal{K}_j(\mathcal{Q}^{(j-1)})|/|\mathcal{K}_j(\mathcal{Q}^{(j-1)})|$*, when* $\mathcal{K}_j(\mathcal{Q}^{(j-1)}) \ne \emptyset$*, and 0 otherwise.*
2. *A* (j,j)*-cylinder* $\mathcal{H}^{(j)}$ *is* (ε,d)-regular *w.r.t. an underlying* $(j,j-1)$*-cylinder* $\mathcal{H}^{(j-1)}$ *if, whenever* $\mathcal{Q}^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$ *satisfies* $\left|\mathcal{K}_j(\mathcal{Q}^{(j-1)})\right| \ge \varepsilon\left|\mathcal{K}_j(\mathcal{H}^{(j-1)})\right|$*, then* $d(\mathcal{H}^{(j)}|\mathcal{Q}^{(j-1)}) = d \pm \varepsilon$.
3. *An* (ℓ,j)*-cylinder* $\mathcal{H}^{(j)}$ *is* (ε,d)-regular *w.r.t. an underlying* $(\ell,j-1)$*-cylinder* $\mathcal{H}^{(j-1)}$ *if, for every* $\Lambda \in \binom{[\ell]}{j}$, $\mathcal{H}^{(j)}[\bigcup_{i\in\Lambda} V_i]$ *is* (ε,d)*-regular w.r.t.* $\mathcal{H}^{(j-1)}[\bigcup_{i\in\Lambda} V_i]$.
4. *An* (ℓ,h)*-complex* $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^h$ *is* $(\varepsilon,(d_2,\dots,d_h))$-regular *if, for every* $j = 2,\dots,h$, $\mathcal{H}^{(j)}$ *is* (ε,d_j)*-regular w.r.t.* $\mathcal{H}^{(j-1)}$.

We prove our following main result in Section 2.

Theorem 1.5 (Grabbing Lemma for Complexes)**.** *For all integers* $k \ge 2$ *and constants* $d_2,\dots,d_{k-1},\varepsilon' > 0$*, there exist* $\varepsilon = \varepsilon_{\text{Thm. 1.5}} > 0$ *and* $c = c_{\text{Thm. 1.5}} > 0$ *and integers* $s_0 = s_{\text{Thm. 1.5}}$ *and* $m_0 = m_{\text{Thm. 1.5}}$ *so that, whenever* $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ *is an* $(\varepsilon,(d_2,\dots,d_{k-1},d_k))$*-regular* (k,k)*-complex, where* $d_k \in [0,1]$ *and* $|V_i| = m_i > m_0$ *for* $1 \le i \le k$*, then, for all* $s_0 \le s \le m_1$*, all but* $\exp\{-c\,s\}\binom{m_1}{s}$ *sets* $S \in \binom{V_1}{s}$ *yield* $\boldsymbol{S} = (S,V_2,\dots,V_k)$ *for which* $\boldsymbol{\mathcal{H}}[\boldsymbol{S}] \overset{\text{def}}{=} \{\mathcal{H}^{(j)}[\boldsymbol{S}]\}_{j=1}^k$ *is an* $(\varepsilon',(d_2,\dots,d_k))$*-regular* (k,k)*-complex.*

We conclude this introduction with two facts used throughout this paper. The first fact is the well-known Chernoff-Hoeffding inequality (see [6]):

If a random variable X *has hypergeometric distribution and* $\varepsilon \in (0,3/2]$*, then*

$$\Pr\left(|X - \mathbb{E}\,X| \ge \varepsilon\,\mathbb{E}\,X\right) \le 2\exp\left\{-\tfrac{\varepsilon^2}{3}\,\mathbb{E}\,X\right\}. \quad (1)$$

The second fact is a 'warm-up' to Theorem 1.3, and asserts that induced subhypergraphs typically inherit correct density.

Fact 1.6. *Let* $\eta > 0$ *be given and suppose* $\mathcal{G}^{(k)}$ *is a* ρ*-dense k-partite k-graph with k-partition* $U_1 \cup \dots \cup U_k$*, where each* $|U_i| = n_i > n_0(\eta)$*. For* $24/\eta^{10} \le s \le n_1$*, all but* $\exp\{-(\eta^8/6)s\}\binom{n_1}{s}$ *sets* $S \in \binom{U_1}{s}$ *render* $\boldsymbol{S} = (S,U_2,\dots,U_k)$ *for which* $d_{\mathcal{G}^{(k)}}(\boldsymbol{S}) = \rho \pm \eta$.

Proof. Without loss of generality, take $0 < \eta < 1/8$ to satisfy $1/\eta^2 \notin \mathbb{N}$. For a vertex $u \in U_1$, let $\mathcal{G}_u^{(k)} = \{K \setminus \{u\} : u \in K \in \mathcal{G}^{(k)}\}$. For $i \in I = [\lceil 1/\eta^2 \rceil]$, let $U_1^i = \{u \in U_1 : (i-1)\eta^2 \leq d_{\mathcal{G}_u^{(k)}}(U_2, \ldots, U_k) < i\eta^2\}$ and write $I^+ = \{i \in I : |U_1^i| \geq \eta^4 n_1\}$ and $I^- = I \setminus I^+$. Then $(\rho - 2\eta^2)n_1 \leq \sum_{i\in I^+} \sum_{u \in U_1^i} d_{\mathcal{G}_u^{(k)}}(U_2, \ldots, U_k) \leq \rho n_1$ and so $\eta^2 \sum_{i \in I^+} i|U_1^i| = (\rho \pm 2\eta^2)n_1$. For $S \in \binom{U_1}{s}$ selected uniformly at random and $i \in I^+$, the Chernoff-Hoeffding inequality (1) ensures

$$\mathbb{P}\big[\exists\, i \in I^+ : |S \cap U_1^i| \neq \big(1 \pm \eta^2\big)\tfrac{s}{n_1}|U_1^i|\big] \leq 2\lceil \eta^{-2} \rceil \exp\big\{-\tfrac{\eta^4}{3}\tfrac{s}{n_1}|U_1^i|\big\} \leq \tfrac{4}{\eta^2}\exp\big\{-\tfrac{\eta^8}{3}s\big\} \leq \exp\big\{-\tfrac{\eta^8}{6}s\big\}.$$

Consider the event that for each $i \in I^+$, $|S \cap U_1^i| = (1 \pm \eta^2)\frac{s}{n_1}|U_1^i|$. Then $d_{\mathcal{G}^{(k)}}(\boldsymbol{S})$ is at least

$$\frac{1}{s}\sum_{i \in I^+} \sum_{u \in S \cap U_1^i} d_{\mathcal{G}_u^{(k)}}(U_2, \ldots, U_k) \geq \frac{\eta^2}{s}\sum_{i\in I^+}(i-1)|S \cap U_1^i| \geq (1-\eta^2)(\rho - 2\eta^2) - \eta^2 \geq \rho - 4\eta^2 > \rho - \eta,$$

and similarly, $d_{\mathcal{G}^{(k)}}(\boldsymbol{S}) \leq (1+\eta^2)(\rho + 2\eta^2) + s^{-1}\sum_{i \in I^-}|S \cap U_1^i|$. Since $\sum_{i\in I^-}|S \cap U_1^i| = s - \sum_{i \in I^+}|S \cap U_1^i| \leq s - (1-\eta^2)\frac{s}{n_1}\sum_{i\in I^+}|U_1^i| \leq 3\eta^2 s$, we obtain $d_{\mathcal{G}^{(k)}}(\boldsymbol{S}) \leq \rho + 8\eta^2 < \rho + \eta$. □

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2. Proof of Theorem 1.5

Notice that, in Theorem 1.5, the constant $d_k \in [0,1]$ is quantified after $d_2, \ldots, d_{k-1} \in (0,1]$ (and, allowed to be zero). We consider the following analogous statement where all $d_2, \ldots, d_{k-1}, d_k > 0$ are quantified together, and up front.

Theorem 2.1. *For all integers $k \geq 2$ and constants $d_2, \ldots, d_{k-1}, d_k, \varepsilon' > 0$, there exist $\varepsilon = \varepsilon_{\text{Thm. 2.1}} > 0$ and $c = c_{\text{Thm. 2.1}} > 0$ and integers $s_0 = s_{\text{Thm. 2.1}}$ and $m_0 = m_{\text{Thm. 2.1}}$ so that, whenever $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ is an $(\varepsilon, (d_2, \ldots, d_{k-1}, d_k))$-regular (k,k)-complex, where $|V_i| = m_i > m_0$ for $1 \leq i \leq k$, then, for all $s_0 \leq s \leq m_1$, all but $\exp\{-c\, s\}\binom{m_1}{s}$ sets $S \in \binom{V_1}{s}$ yield $\boldsymbol{S} = (S, V_2, \ldots, V_k)$ for which $\boldsymbol{\mathcal{H}}[\boldsymbol{S}] \stackrel{\text{def}}{=} \{\mathcal{H}^{(j)}[\boldsymbol{S}]\}_{j=1}^k$ is an $(\varepsilon', (d_2, \ldots, d_k))$-regular (k,k)-complex.*

Clearly, Theorem 1.5 implies Theorem 2.1, but these statements are, in fact, equivalent. We establish that Theorem 2.1 implies Theorem 1.5 in the Appendix. In the remainder of this section, we prove Theorem 2.1. To that end, our proof takes place in three steps. In Section 2.1, we prove Theorem 2.1 for $k = 2$. In Section 2.2, we show that the case $k = 2$ implies the case when, in the complex $\boldsymbol{\mathcal{H}}$, $\mathcal{H}^{(k-1)} = K^{(k-1)}[V_1, \ldots, V_\ell]$. In Section 2.3, we show the latter case implies Theorem 2.1 in full.

2.1. **Proof when $k = 2$.** The proof of Theorem 2.1 when $k = 2$ is well-known (and short). We include it here for completeness, and to that end, use the following lemma of Alon et al. [2] (adapted from [3]).

Lemma 2.2. *Let $d > 0$ be given and let F be a bipartite graph with bipartition $X \cup Y$. If $0 < 4\mu < d^2$ and F is (μ, d)-regular (w.r.t. $X \cup Y$), then all but $2\mu|X|$ vertices $x \in X$ satisfy $\deg(x) = (d \pm \mu)|Y|$, and all but $4\mu|X|^2$ pairs $x, x' \in X$ satisfy $\deg(x, x') = (d \pm \mu)^2|Y|$. Conversely, if $|X|, |Y|$ are sufficiently large w.r.t. d, μ and all but $\mu|X|$ vertices $x \in X$ have $\deg(x) = (d \pm \mu)|Y|$ and all but $\mu|X|^2$ pairs $x, x' \in X$ have $\deg(x, x') = (d \pm \mu)^2|Y|$, then F is $(3\mu^{1/5}, d)$-regular.*

Proof of Theorem 2.1 ($k = 2$). Let $d_2, \varepsilon' > 0$ be given. Set $\varepsilon = d_2^2(\varepsilon'/5)^5$ and $c = (\varepsilon d_2)^2/24$. Let $s_0 \geq 96/(\varepsilon^3 d_2^2)$ be large enough (as a lower bound on $|X|, |Y|$) to enable an application of Lemma 2.2. We take m_1, m_2 sufficiently large whenever needed. Let $H = \mathcal{H}^{(2)}$ be an (ε, d_2)-regular bipartite graph with bipartition $\mathcal{H}^{(1)} = V_1 \cup V_2$, where $|V_1| = m_1$ and $|V_2| = m_2$. For $s_0 \leq s \leq m_1$, let $S \in \binom{V_1}{s}$ be chosen uniformly at random. For vertices $v_2, v_2' \in V_2$, write $\deg_S(v_2) = |N_H(v_2) \cap S|$ and $\deg_S(v_2, v_2') = |N_H(v_2, v_2') \cap S|$, where $N_H(v_2, v_2') = N_H(v_2) \cap N_H(v_2')$. Let V_2' be the set of vertices $v_2 \in V_2$ for which $\deg(v_2) = (d_2 \pm \varepsilon)m_1$ and let $\binom{V_2}{2}'$ be the set of pairs $\{v_2, v_2'\} \in \binom{V_2}{2}$ for which $\deg(v_2, v_2') = (d_2 \pm \varepsilon)^2 m_1$. For fixed $v_2 \in V_2'$, the Chernoff-Hoeffding inequality (1) gives

$$\mathbb{P}\big[\deg_S(v_2) \neq (d_2 \pm 3\varepsilon)s\big] \leq \mathbb{P}\big[\deg_S(v_2) \neq (1 \pm \varepsilon)\mathbb{E}\deg_S(v_2)\big] \leq 2\exp\big\{-\tfrac{\varepsilon^2}{3}(d_2 - \varepsilon)s\big\} \leq 2\exp\big\{-\tfrac{\varepsilon^2}{6}d_2 s\big\}.$$

Similarly, for fixed $\{v_2, v_2'\} \in \binom{V_2}{2}'$, we have $\mathbb{P}[\deg_S(v_2, v_2') \neq (d_2 \pm 3\varepsilon)^2 s] \leq 2\exp\{-(\varepsilon^2/12)d_2^2 s\}$. Now, define $V_{2,S}' \subseteq V_2'$ to be the set of vertices $v_2 \in V_2'$ for which $\deg_S(v_2) \neq (d_2 \pm 3\varepsilon)s$ and define $\binom{V_2}{2}_S' \subseteq \binom{V_2}{2}'$ to be the set of pairs $\{v_2, v_2'\} \in \binom{V_2}{2}'$ for which $\deg_S(v_2, v_2') \neq (d_2 \pm 3\varepsilon)^2 s$. By the Markov inequality,

$$\mathbb{P}\big[|V_{2,S}'| \geq \varepsilon m_2 \text{ or } \big|\tbinom{V_2}{2}_S'\big| \geq \varepsilon m_2^2\big] \leq \tfrac{2}{\varepsilon}\exp\big\{-\tfrac{\varepsilon^2}{6}d_2 s\big\} + \tfrac{2}{\varepsilon}\exp\big\{-\tfrac{\varepsilon^2}{12}d_2^2 s\big\} \leq \exp\big\{-\tfrac{\varepsilon^2}{24}d_2^2 s\big\} = \exp\big\{-cs\big\}.$$

The event $|V_{2,S}'| < \varepsilon m_2$ and $\big|\binom{V_2}{2}_S'\big| < \varepsilon m_2^2$ implies $H[S, V_2]$ is (ε', d_2)-regular. Indeed, by Lemma 2.2, $|V_2 \setminus V_2'| < 2\varepsilon m_2$ so that with $|V_{2,S}'| < \varepsilon m_2$, we have all but $3\varepsilon m_2$ vertices $v_2 \in V_2$ satisfying $\deg_S(v_2) = (d_2 \pm 3\varepsilon)s$. By Lemma 2.2, $\big|\binom{V_2}{2} \setminus \binom{V_2}{2}'\big| < 4\varepsilon m_2^2$ so that with $\big|\binom{V_2}{2}_S'\big| < \varepsilon m_2^2$, we have all but $5\varepsilon m_2^2$ pairs $\{v_2, v_2'\} \in \binom{V_2}{2}$ satisfying $\deg_S(v_2, v_2') = (d_2 \pm 3\varepsilon)^2 s$. As such, Lemma 2.2 says that $H[S, V_2]$ is $(3(5\varepsilon)^{1/5}, d_2)$-regular, and so is (ε', d_2)-regular. □

2.2. **Proof for complete underlying cylinders.** We use the following lemma of Kohayakawa, Rödl and Skokan [7], which is an extension of Lemma 2.2. Let $\mathcal{F}^{(j)}$ be a (j,j)-cylinder with $V(\mathcal{F}^{(j)}) = X_1 \cup \cdots \cup X_j$. For $x, y \in V(\mathcal{F}^{(j)})$, let $\mathcal{F}_x^{(j)} = \{J \setminus \{x\} : x \in J \in \mathcal{F}^{(j)}\}$ and $\mathcal{F}_{xy}^{(j)} = \mathcal{F}_x^{(j)} \cap \mathcal{F}_y^{(j)}$. Let $K_{2,j}^{(j)}$ denote the complete j-partite j-graph with 2 vertices in each class and let $\mathcal{K}_{2,j}(\mathcal{F}^{(j)})$ denote the family of all $(2j)$-element subsets of $V(\mathcal{F}^{(j)})$ which span a copy of $K_{2,j}^{(j)}$ in $\mathcal{F}^{(j)}$. Lemma 2.3 establishes the equivalence of the following three statements:

$\mathbf{S}_1(d, \eta_1)$: $\mathcal{F}^{(j)}$ is (η_1, d)-regular (w.r.t. $K^{(j-1)}[X_1, \ldots, X_j]$).

$\mathbf{S}_2(d, \eta_2)$: All but $\eta_2|X_1|$ vertices $x \in X_1$ satisfy that $\mathcal{F}_x^{(j)}$ is (η_2, d)-regular (w.r.t. $K^{(j-2)}[X_2, \ldots, X_j]$) and all but $\eta_2|X_1|^2$ pairs $x, x' \in X_1$ satisfy that $\mathcal{F}_{xx'}^{(j)}$ is (η_2, d^2)-regular.

$\mathbf{S}_3(d, \eta_3)$: $\mathcal{F}^{(j)}$ has density $d_{\mathcal{F}^{(j)}}(X_1, \ldots, X_j) = d \pm \eta_3$ and $\big|\mathcal{K}_{2,j}(\mathcal{F}^{(j)})\big| = (1 \pm \eta_3)(d \pm \eta_3)^{2^j} \prod_{i=1}^{j} \binom{|X_i|}{2}$.

Statements $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ are equivalent in the following sense.

Lemma 2.3 (Kohayakawa, Rödl, Skokan [7]). *Let $j \geq 2$ be an integer, let $d > 0$ be given and fix $1 \leq a, b \leq 3$. For all $\eta_a > 0$, there exists $\eta_b > 0$ so that whenever $\mathcal{F}^{(j)}$ is a (j,j)-cylinder (as above) with each $|X_1|, \ldots, |X_j|$ sufficiently large, then, if $\mathcal{F}^{(j)}$ satisfies $\mathbf{S}_b(d, \eta_b)$, then it also satisfies $\mathbf{S}_a(d, \eta_a)$.*

Proof of Theorem 2.1 (complete underlying cylinders). Let integer $k \geq 3$, $d_k, \varepsilon' > 0$ be given. We define the promised constants ε, c, s_0 in terms of auxiliary constants (and provide a summary of constants below in (2)). Let $\eta_7 = \eta_{\text{Lem. 2.3}}(d_k, \varepsilon')$ be the constant guaranteed by Lemma 2.3 to satisfy $\mathbf{S}_3(d_k, \eta_7) \implies \mathbf{S}_1(d_k, \varepsilon')$ (with $j = k$). Set

$$\eta_6 = \tfrac{1}{5}\min\big\{\eta_7^{2^k}, d_k^{2^k} - (d_k - \eta_7)^{2^k}\big\}$$

and let $\eta_5 = \eta_{\text{Thm. 2.1},k=2}(d_k^{2^{k-1}}, \eta_6)$, $c_* = c_{\text{Thm. 2.1},k=2}(d_k^{2^{k-1}}, \eta_6, \eta_5)$ and $s_* = s_{\text{Thm. 2.1},k=2}(d_k^{2^{k-1}}, \eta_6, \eta_5)$ be the constants guaranteed to exist by Theorem 2.1 ($k = 2$). (As we shall use, the proof of Theorem 2.1 ($k = 2$) gives $c_* = d_k^{2^k}\eta_5^2/24$.) Set $\eta_4 = (\eta_5/3)^5$ and $\eta_3 = d_k^2 2^{-2^k}\eta_4$. Let $\eta_2 = \eta_{\text{Lem. 2.3}}(d_k, \eta_3)$ be the constant guaranteed by Lemma 2.3 to satisfy both $\mathbf{S}_1(d_k, \eta_2) \implies \mathbf{S}_3(d_k, \eta_3)$ and $\mathbf{S}_1(d_k^2, \eta_2) \implies \mathbf{S}_3(d_k^2, \eta_3)$ (with $j = k-1$). Let $\eta_1 = \eta_{\text{Lem. 2.3}}(d_k, \eta_2)$ be the constant guaranteed by Lemma 2.3 to satisfy $\mathbf{S}_1(d_k, \eta_1) \implies \mathbf{S}_2(d_k, \eta_2)$ (with $j = k$). Define $\varepsilon = \eta_1$, $c = c_*/2$ and $s_0 = \max\{s_*, 4/c, 2/(d_k^{2^k}\eta_7), 24/\eta_6^{10}\}$. Note that all constants above can be summarized by the following hierarchy:

$$d_k,\, \varepsilon' \gg \eta_7 \gg \eta_6 \gg \eta_5 \gg \begin{cases} \eta_4 = \left(\frac{\eta_5}{3}\right)^5 \gg \eta_3 = d_k^2 2^{-2^k}\eta_4 \gg \eta_2 \gg \eta_1 = \varepsilon \\ \max\left\{c_* = \frac{d_k^{2^k}\eta_5^2}{24} = 2c, \frac{1}{s_*}\right\} \geq \min\left\{c_*, \frac{1}{s_*}\right\} \geq \frac{1}{s_0}. \end{cases} \tag{2}$$

We take m_0 sufficiently large with respect to all constants above whenever needed. Now, let $\mathcal{H}^{(k)}$ be a (k,k)-cylinder with vertex partition $V_1 \cup \cdots \cup V_k$, where for each $1 \leq i \leq k$, $|V_i| = m_i \geq m_0$, and where $\mathcal{H}^{(k)}$ is (ε, d_k)-regular (w.r.t. $K^{(k-1)}[V_1, \ldots, V_k]$). For given $s_0 \leq s \leq m_1$, we show all but $\exp\{-cs\}\binom{m_1}{s}$ sets $S \in \binom{V_1}{s}$ yield $\boldsymbol{S} = (S, V_2, \ldots, V_k)$ for which $\mathcal{H}^{(k)}[\boldsymbol{S}]$ is (ε', d_k)-regular.

We use the following auxiliary bipartite graph F with bipartition $V(F) = X \cup Y$, where $X = V_1$ and $Y = \mathcal{K}_{2,k-1}(K^{(k-1)}[V_2, \ldots, V_k])$. Note that $|X| = m_1$ and $|Y| = \prod_{i=2}^{k}\binom{m_i}{2}$. For $x \in X$ and $y \in Y$, let

$\{x,y\} \in F$ if and only if $y \in \mathcal{K}_{2,k-1}(\mathcal{H}_x^{(k)})$. We claim that F is $(\eta_5, d_k^{2^{k-1}})$-regular, and more strongly (cf. Lemma 2.2) that

$$\text{all but } \eta_4|X| \text{ vertices } x \in X \text{ satisfy } \deg_F(x) = \big(d_k^{2^{k-1}} \pm \eta_4\big)|Y|$$
$$\text{and all but } \eta_4|X|^2 \text{ pairs } \{x,x'\} \in \tbinom{X}{2} \text{ satisfy } \deg_F(x,x') = \big(d_k^{2^{k-1}} \pm \eta_4\big)^2|Y|. \quad (3)$$

To see (3), observe first that for each $\{x,x'\} \in \binom{X}{2}$, $\deg_F(x) = \big|\mathcal{K}_{2,k-1}(\mathcal{H}_x^{(k)})\big|$ and $\deg_F(x,x') = \big|\mathcal{K}_{2,k-1}(\mathcal{H}_{xx'}^{(k)})\big|$, where these quantities can be estimated with Lemma 2.3. Indeed, since $\mathcal{H}^{(k)}$ is (ε, d_k)-regular, Lemma 2.3 ($\mathbf{S}_1 \implies \mathbf{S}_2$) asserts that all but $\eta_2|V_1|$ vertices $x \in V_1$ satisfy that $\mathcal{H}_x^{(k)}$ is (η_2, d_k)-regular, and for a fixed such $x \in V_1$, Lemma 2.3 ($\mathbf{S}_1 \implies \mathbf{S}_3$) also asserts that

$$\big|\mathcal{K}_{2,k-1}(\mathcal{H}_x^{(k)})\big| = (1 \pm \eta_3)(d_k \pm \eta_3)^{2^{k-1}} \prod_{i=2}^{k} \tbinom{m_i}{2} = \big(d_k^{2^{k-1}} \pm \eta_4\big)|Y|.$$

This establishes the first assertion in (3), and an analogous argument establishes the second one.

Since F is $(\eta_5, d_k^{2^{k-1}})$-regular, Theorem 2.1 ($k=2$) ensures that all but $\exp\{-c_* s\}\binom{m_1}{s}$ sets $S \in \binom{X}{s} = \binom{V_1}{s}$ satisfy that $F[S,Y]$ is $(\eta_6, d_k^{2^{k-1}})$-regular. For $\boldsymbol{V} = (V_1, \ldots, V_k)$, set $d = d_{\mathcal{H}^{(k)}}(\boldsymbol{V})$ so that $d = d_k \pm \varepsilon$. Fact 1.6 ensures that all but $\exp\{-(\eta_6^8/6)s\}\binom{m_1}{s}$ sets $S \in \binom{V_1}{s}$ satisfy that $d_{\mathcal{H}^{(k)}}(\boldsymbol{S}) = d \pm \eta_6 = d_k \pm \eta_7$. Fix a set S satisfying both conditions, noting that a proportion of at most $\exp\{-c_* s\} + \exp\{-(\eta_6^8/6)s\} \le \exp\{-\frac{c_*}{2}s\} = \exp\{-cs\}$ sets of $\binom{X}{s}$ would not. Since $F[S,Y]$ is $(\eta_6, d_k^{2^{k-1}})$-regular, Lemma 2.2 says that all but $2\eta_6|Y|$ vertices $y \in Y$ satisfy $\deg_S(y) = (d_k^{2^{k-1}} \pm \eta_6)s$. The construction of the graph F ensures $|\mathcal{K}_{2,k}(\mathcal{H}^{(k)}[\boldsymbol{S}])| = \sum_{y \in Y} \binom{\deg_S(y)}{2}$, and so

$$\big|\mathcal{K}_{2,k}(\mathcal{H}^{(k)}[\boldsymbol{S}])\big| \le \big(d_k^{2^{k-1}} + \eta_6\big)^2 \tfrac{s^2}{2}|Y| + \eta_6 s^2|Y| \le \big(d_k^{2^k} + 5\eta_6\big)\big(1 + \tfrac{1}{s-1}\big)\tbinom{s}{2}|Y|$$
$$\le (1+\eta_7)\big(d_k^{2^k} + \eta_7^{2^k}\big)\tbinom{s}{2}|Y| \le (1+\eta_7)(d_k + \eta_7)^{2^k}\tbinom{s}{2}\prod_{i=2}^{k}\tbinom{m_i}{2}, \quad \text{and}$$

$$\big|\mathcal{K}_{2,k}(\mathcal{H}^{(k)}[\boldsymbol{S}])\big| \ge \big(d_k^{2^{k-1}} - \eta_6\big)^2 \tfrac{s^2}{2}\Big(1 - \tfrac{1}{(d_k^{2^{k-1}} - \eta_6)s}\Big)(|Y| - 2\eta_6|Y|) \ge (1-\eta_7)\big(d_k^{2^k} - 4\eta_6\big)\tbinom{s}{2}|Y|$$
$$\ge (1-\eta_7)(d_k - \eta_7)^{2^k}\tbinom{s}{2}\prod_{i=2}^{k}\tbinom{m_i}{2} \quad \implies \quad \big|\mathcal{K}_{2,k}(\mathcal{H}^{(k)}[\boldsymbol{S}])\big| = (1 \pm \eta_7)(d_k \pm \eta_7)^{2^k}\tbinom{s}{2}\prod_{i=2}^{k}\tbinom{m_i}{2}.$$

But now, $\mathcal{H}^{(k)}[\boldsymbol{S}]$ has density $d_{\mathcal{H}^{(k)}}(\boldsymbol{S}) = d_k \pm \eta_7$ and satisfies the estimates above so that Lemma 2.3 ($\mathbf{S}_3 \implies \mathbf{S}_1$) implies that $\mathcal{H}^{(k)}[\boldsymbol{S}]$ is (ε', d_k)-regular. □

2.3. **Proof of Theorem 2.1.** We now prove that Theorem 2.1 follows from the special case of the previous subsection. A tool in our argument is the following 'dense counting lemma' of Kohayakawa, Rödl and Skokan (Theorem 6.5 in [7]).

Lemma 2.4 (Dense Counting Lemma). *For all integers $\ell \ge j \ge 2$ and $\gamma, d_j, \ldots, d_2 > 0$, there exist an $\varepsilon = \varepsilon_{\text{Lem. 2.4}} > 0$ and a positive integer $m_0 = m_{\text{Lem. 2.4}}$ so that whenever $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(h)}\}_{h=1}^{j}$ is an $(\varepsilon, (d_2, \ldots, d_j))$-regular (ℓ, j)-complex with $|V_i| = m_i > m_0$ for $1 \le i \le \ell$, then $|\mathcal{K}_\ell(\mathcal{H}^{(j)})| = (1 \pm \gamma)\prod_{h=2}^{j} d_h^{\binom{\ell}{h}} \times \prod_{i=1}^{\ell} m_i$.*

Proof of Theorem 2.1. Let integer $k \ge 3$ and $d_k, d_{k-1}, \ldots, d_2, \varepsilon' > 0$ be given. Without loss of generality, assume $\varepsilon' < \frac{1}{2}\prod_{h=2}^{k-1} d_h^{\binom{k-1}{h}}$. We define the promised constants ε, c, s_0 in terms of auxiliary constants. For $3 \le j \le k$, let $\hat{\varepsilon}_j = \varepsilon_{\text{Lem. 2.4}}(j, j-1, 1/2, d_{j-1}, \ldots, d_2)$ and $\hat{m}_j = m_{\text{Lem. 2.4}}(j, j-1, 1/2, d_{j-1}, \ldots, d_2)$ be the constants guaranteed by Lemma 2.4. We shall assume, w.l.o.g., that $2\varepsilon' \le \min\{\hat{\varepsilon}_3, \ldots, \hat{\varepsilon}_k, \prod_{j=2}^{k-1} d_j^{k^j}\}$. For $2 \le j \le k$, let $\varepsilon_j = \varepsilon_{\text{Thm. 2.1, comp}}(d_j, (\varepsilon')^2)$, $c_j = c_{\text{Thm. 2.1, comp}}(d_j, (\varepsilon')^2)$

and $s_j = s_{\text{Thm. 2.1, comp}}(d_j, (\varepsilon')^2)$ be the constants guaranteed by Theorem 2.1 (for complete underlying cylinders). Set

$$\varepsilon = \tfrac{1}{2}\min\{\varepsilon_2^2, \dots, \varepsilon_k^2\}, c = \tfrac{1}{2}\min\{c_2, \dots, c_k\} \text{ and } s_0 = \max\{s_2, \dots, s_k, \hat{m}_3, \dots, \hat{m}_k, 2k/c\}.$$

We shall take m_0 sufficiently large whenever needed. With these constants, let $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ be an $(\varepsilon, (d_2, \dots, d_k))$-regular (k,k)-complex, where $|V_i| = m_i > m_0$ for each $1 \le i \le k$. Let $s_0 \le s \le m_1$ be given. We prove that all but $\exp\{-cs\}\binom{m_1}{s}$ sets $S \in \binom{V_1}{s}$ yield $\boldsymbol{S} = (S, V_2, \dots, V_k)$ for which $\boldsymbol{\mathcal{H}}[\boldsymbol{S}]$ is an $(\varepsilon', (d_2, \dots, d_k))$-regular (k,k)-complex.

We prove, by induction on $2 \le j \le k$, that for each choice of indices $2 \le i_2 < \dots < i_j \le k$,

all but $\binom{m_1}{s}\sum_{i=2}^{j}\binom{j-1}{i-1}\exp\{-c_i s\}$ sets $S \in \binom{V_1}{s}$ satisfy that

$$\boldsymbol{\mathcal{H}}^{(j)}[S, V_{i_2}, \dots, V_{i_j}] \text{ is an } (\varepsilon', (d_2, \dots, d_j))\text{-regular } (j,j)\text{-complex,} \quad (4)$$

where $\boldsymbol{\mathcal{H}}^{(j)} = \{\mathcal{H}^{(h)}\}_{h=1}^j$. Theorem 2.1 then easily follows from (4) with $j = k$. Note that (4) for $j = 2$ holds on account of the first subsection. Now, fix indices $2 \le i_2 < \dots < i_j \le k$, w.l.o.g., $i_2 = 2, \dots, i_j = j$. If (4) holds through $2 \le j-1 < k$, then all but

$$\binom{m_1}{s}\sum_{i=2}^{j-1}\binom{j-2}{i-1}\binom{j-1}{j-i}\exp\{-c_i s\} = \binom{m_1}{s}\sum_{i=2}^{j-1}\binom{j-1}{i-1}\exp\{-c_i s\}$$

sets $S \in \binom{V_1}{s}$ yield $\boldsymbol{S} \overset{\text{def}}{=} (S, V_2, \dots, V_j)$ for which $\boldsymbol{\mathcal{H}}^{(j-1)}[\boldsymbol{S}] = \{\mathcal{H}^{(h)}[\boldsymbol{S}]\}_{h=1}^{j-1}$ is an $(\varepsilon', (d_2, \dots, d_{j-1}))$-regular $(j, j-1)$-complex. Let us denote the collection of these sets S by $\binom{V_1}{s}_{<j}$. Verifying (4) then reduces to showing that

$$\text{all but } \exp\{-c_j s\}\binom{m_1}{s} \text{ sets } S \in \binom{V_1}{s}_{<j} \quad (5)$$

yield $\boldsymbol{S} = (S, V_2, \dots, V_j)$ for which $\mathcal{H}^{(j)}[\boldsymbol{S}]$ is (ε', d_j)-regular w.r.t. $\mathcal{H}^{(j-1)}[\boldsymbol{S}]$. To that end, write $\mathcal{H}^{(h)}$ in place of $\mathcal{H}^{(h)}[V_1, \dots, V_j]$, $1 \le h \le j$, so that $\boldsymbol{\mathcal{H}}^{(j)} = \{\mathcal{H}^{(h)}\}_{h=1}^j$ is an $(\varepsilon, (d_2, \dots, d_j))$-regular (j,j)-complex ($\mathcal{H}^{(1)} = V_1 \cup \dots \cup V_j$) satisfying that all sets $S \in \binom{V_1}{s}_{<j}$ yield $\boldsymbol{S} = (S, V_2, \dots, V_j)$ for which $\boldsymbol{\mathcal{H}}^{(j-1)}[\boldsymbol{S}]$ is an $(\varepsilon', (d_2, \dots, d_{j-1}))$-regular $(j, j-1)$-complex. We make the following claim.

Claim 2.5. *There exists a (j,j)-cylinder $\tilde{\mathcal{H}}^{(j)}$ with vertex partition $\boldsymbol{V} = (V_1, \dots, V_j)$ which is $(2\varepsilon^{1/2}, d_j)$-regular w.r.t. $K^{(j-1)}[\boldsymbol{V}]$ and for which $\mathcal{H}^{(j)} = \tilde{\mathcal{H}}^{(j)} \cap \mathcal{K}_j(\mathcal{H}^{(j-1)})$.*

Now, by Theorem 2.1 (complete underlying cylinders), all but $\exp\{-c_j s\}\binom{m_1}{s}$ sets $S \in \binom{V_1}{s}_{<j}$ yield $\tilde{\mathcal{H}}^{(j)}[\boldsymbol{S}]$ which is $((\varepsilon')^2, d_j)$-regular (w.r.t. $K^{(j-1)}[\boldsymbol{S}]$). We claim that any such $S \in \binom{V_1}{s}_{<j}$ also satisfies that $\mathcal{H}^{(j)}[\boldsymbol{S}]$ is (ε', d_j)-regular w.r.t. $\mathcal{H}^{(j-1)}[\boldsymbol{S}]$. Indeed, fix such an $S \in \binom{V_1}{s}_{<j}$ and let $\mathcal{Q}^{(j-1)} \subseteq \mathcal{H}^{(j-1)}[\boldsymbol{S}] \subseteq K^{(j-1)}[\boldsymbol{S}]$ satisfy $|\mathcal{K}_j(\mathcal{Q}^{(j-1)})| > \varepsilon'|\mathcal{K}_j(\mathcal{H}^{(j-1)}[\boldsymbol{S}])|$. Lemma 2.4, implies

$$\left|\mathcal{K}_j(\mathcal{H}^{(j-1)}[\boldsymbol{S}])\right| \ge \tfrac{1}{2}\prod_{h=2}^{j-1} d_h^{\binom{j}{h}} \times s\prod_{i=2}^{j} m_i > \varepsilon' s\prod_{i=2}^{j} m_i \quad \Longrightarrow \quad \left|\mathcal{K}_j(\mathcal{Q}^{(j-1)})\right| > (\varepsilon')^2 s\prod_{i=2}^{j} m_i.$$

Since $\tilde{\mathcal{H}}^{(j)}[\boldsymbol{S}]$ is $((\varepsilon')^2, d_j)$-regular, we have $|\tilde{\mathcal{H}}^{(j)}[\boldsymbol{S}] \cap \mathcal{K}_j(\mathcal{Q}^{(j-1)})| = (d_j \pm \varepsilon')|\mathcal{K}_j(\mathcal{Q}^{(j-1)})|$. Since $\mathcal{Q}^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$, Claim 2.5 implies $\tilde{\mathcal{H}}^{(j)}[\boldsymbol{S}] \cap \mathcal{K}_j(\mathcal{Q}^{(j-1)}) = \mathcal{H}^{(j)}[\boldsymbol{S}] \cap \mathcal{K}_j(\mathcal{Q}^{(j-1)})$, which proves (5). □

Proof of Claim 2.5. We define $\tilde{\mathcal{H}}^{(j)}$ by adding to $\mathcal{H}^{(j)}$ each j-tuple $J \in K^{(j)}[\boldsymbol{V}] \setminus \mathcal{K}_j(\mathcal{H}^{(j-1)})$ independently with probability d_j. Clearly, $\mathcal{H}^{(j)} = \tilde{\mathcal{H}}^{(j)} \cap \mathcal{K}_j(\mathcal{H}^{(j-1)})$. Standard details now show that, w.h.p., $\tilde{\mathcal{H}}^{(j)}$ is $(2\varepsilon^{1/2}, d_j)$-regular. Indeed, let $\tilde{\mathcal{Q}}^{(j-1)} \subseteq K^{(j-1)}[\boldsymbol{V}]$ satisfy $|\mathcal{K}_j(\tilde{\mathcal{Q}}^{(j-1)})| > 2\varepsilon^{1/2}\prod_{i=1}^{j} m_i \overset{\text{def}}{=} 2\varepsilon^{1/2}M$. Write $\mathcal{Q}^{(j-1)} = \tilde{\mathcal{Q}}^{(j-1)} \cap \mathcal{H}^{(j-1)}$ and observe that $\mathcal{K}_j(\tilde{\mathcal{Q}}^{(j-1)}) \cap \mathcal{K}_j(\mathcal{H}^{(j-1)}) = \mathcal{K}_j(\tilde{\mathcal{Q}}^{(j-1)} \cap \mathcal{H}^{(j-1)}) = \mathcal{K}_j(\mathcal{Q}^{(j-1)})$. Since $\tilde{\mathcal{H}}^{(j)} \cap \mathcal{K}_j(\mathcal{Q}^{(j-1)}) = \mathcal{H}^{(j)} \cap \mathcal{K}_j(\mathcal{Q}^{(j-1)})$, the (ε, d_j)-regularity of $\mathcal{H}^{(j)}$ w.r.t. $\mathcal{H}^{(j-1)}$ implies $|\tilde{\mathcal{H}}^{(j)} \cap \mathcal{K}_j(\mathcal{Q}^{(j-1)})| = (d_j \pm \varepsilon)|\mathcal{K}_j(\mathcal{Q}^{(j-1)})|$ if $|\mathcal{K}_j(\mathcal{Q}^{(j-1)})| > \varepsilon|\mathcal{K}_j(\mathcal{H}^{(j-1)})|$, and is at most εM otherwise. From the Chernoff-Hoeffding inequality (1), we have with probability $1 - \exp\{-\Omega(M/\log M)\}$ that $|\tilde{\mathcal{H}}^{(j)} \cap (\mathcal{K}_j(\tilde{\mathcal{Q}}^{(j-1)}) \setminus \mathcal{K}_j(\mathcal{H}^{(j-1)}))| = (d_j \pm o(1))|\mathcal{K}_j(\tilde{\mathcal{Q}}^{(j-1)}) \setminus \mathcal{K}_j(\mathcal{H}^{(j-1)})|$ if $|\mathcal{K}_j(\tilde{\mathcal{Q}}^{(j-1)}) \setminus \mathcal{K}_j(\mathcal{H}^{(j-1)})| > M/\log M$, and is at most $M/\log M$ otherwise. Altogether, we conclude that with probability $1 - \exp\{-\Omega(M/\log M)\}$ (recall $|\mathcal{K}_j(\tilde{\mathcal{Q}}^{(j-1)})| > 2\varepsilon^{1/2}M$),

$$\big|\tilde{\mathcal{H}}^{(j)} \cap \mathcal{K}_j(\tilde{\mathcal{Q}}^{(j-1)})\big| \le \big(d_j + \varepsilon + o(1)\big)\big|\mathcal{K}_j(\tilde{\mathcal{Q}}^{(j-1)})\big| + \varepsilon M + \tfrac{M}{\log M} \le \big(d_j + 2\varepsilon^{1/2}\big)\big|\mathcal{K}_j(\tilde{\mathcal{Q}}^{(j-1)})\big|, \text{ and}$$
$$\big|\tilde{\mathcal{H}}^{(j)} \cap \mathcal{K}_j(\tilde{\mathcal{Q}}^{(j-1)})\big| \ge \big(d_j - \varepsilon - o(1)\big)\big|\mathcal{K}_j(\tilde{\mathcal{Q}}^{(j-1)})\big| - \varepsilon M - \tfrac{M}{\log M} \ge \big(d_j - 2\varepsilon^{1/2}\big)\big|\mathcal{K}_j(\tilde{\mathcal{Q}}^{(j-1)})\big|.$$

Since there are at most $2^{M\sum_{i=1}^{j} m_i^{-1}} = \exp\{o(M/\log M)\}$ sub-hypergraphs $\tilde{\mathcal{Q}}^{(j-1)} \subseteq K^{(j-1)}[\boldsymbol{V}]$, we conclude that, with probability $1 - o(1)$, $\tilde{\mathcal{H}}^{(j)}$ is $(2\varepsilon^{1/2}, d)$-regular. □

3. Regular-approximation Lemma

In this section, we state a regular-approximation lemma (Theorem 3.4) from [11]. We then state and prove a related proposition (Proposition 3.6).

3.1. **Regular-approximation Lemma.** The regular-approximation lemma for k-uniform hypergraphs provides a well-structured family of partitions $\mathscr{P} = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ of vertices, pairs, …, and $(k-1)$-tuples of a given vertex set V. We describe the form of these partitions inductively (cf. [10, 11]):

(a) Let $\mathscr{P}^{(1)} = \{V_1, \dots, V_{|\mathscr{P}^{(1)}|}\}$ be a partition of V. Relatedly, for $1 \le j \le |\mathscr{P}^{(1)}|$, let
- $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ be the family of all crossing j-tuples J;
- $\mathscr{B}^{(j)}$ be the (auxiliary) partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ with classes $K^{(j)}[V_{i_1}, \dots, V_{i_j}]$, $1 \le i_1 < \cdots < i_j \le |\mathscr{P}^{(1)}|$.

(b) Fix an integer $1 \le j \le k-1$. Assume that, for each $1 \le i \le j-1$, a partition $\mathscr{P}^{(i)}$ of $\mathrm{Cross}_i(\mathscr{P}^{(1)})$ has been defined which refines $\mathscr{B}^{(i)}$. (These partitions will, inductively, satisfy a stronger condition revealed in the inductive step.) Relatedly,
- for each $I \in \mathrm{Cross}_{j-1}(\mathscr{P}^{(1)})$, write $\mathcal{P}^{(j-1)}(I)$ for the unique partition class in $\mathscr{P}^{(j-1)}$ that contains I;
- for each $J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})$, define the *polyad of* J by $\hat{\mathcal{P}}^{(j-1)}(J) = \bigcup\big\{\mathcal{P}^{(j-1)}(I)\colon\ I \in \binom{J}{j-1}\big\}$, which (since $\mathscr{P}^{(j-1)}$ refines $\mathscr{B}^{(j-1)}$) is the union of the unique collection of j distinct partition classes of $\mathscr{P}^{(j-1)}$, each containing a $(j-1)$-subset of J;
- define the family of all polyads $\hat{\mathscr{P}}^{(j-1)} = \big\{\hat{\mathcal{P}}^{(j-1)}(J)\colon\ J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})\big\}$, and view $\hat{\mathscr{P}}^{(j-1)}$ as a set with elements $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$. (In particular, note that $\hat{\mathcal{P}}^{(j-1)}(J)$ and $\hat{\mathcal{P}}^{(j-1)}(J')$ are not necessarily distinct for $J \ne J'$.)

(c) Let $\mathscr{P}^{(j)}$ be a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ which refines the partition $\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\colon\ \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ (which, in turn, refines the partition $\mathscr{B}^{(j)}$). Note, in particular, that
- the set of cliques spanned by a polyad in $\hat{\mathscr{P}}^{(j-1)}$ is sub-partitioned in $\mathscr{P}^{(j)}$;
- every partition class in $\mathscr{P}^{(j)}$ belongs to precisely one polyad in $\hat{\mathscr{P}}^{(j-1)}$.

This concludes our description.

We continue by defining some considerations and notation related to a family $\mathscr{P}$ as described above. First, note that for each $1 \le j \le k-1$ and $J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})$, $\hat{\mathcal{P}}^{(j-1)}(J) \in \hat{\mathscr{P}}^{(j)}$ is a $(j, j-1)$-cylinder. More generally, for $1 \le i < j$, note that $\hat{\mathcal{P}}^{(i)}(J) = \bigcup\big\{\mathcal{P}^{(i)}(I)\colon\ I \in \binom{J}{i}\big\}$ is a (j, i)-cylinder, and therefore, $\boldsymbol{\mathcal{P}}(J) = \{\hat{\mathcal{P}}^{(i)}(J)\}_{i=1}^{j-1}$ is a $(j, j-1)$-complex. When we drop the argument J and write $\hat{\mathcal{P}}^{(j-1)}$ for $\hat{\mathcal{P}}^{(j-1)}(J)$, we shall correspondingly write

$$\boldsymbol{\mathcal{P}}_{\hat{\mathcal{P}}^{(j-1)}} = \boldsymbol{\mathcal{P}}(J)\,. \tag{6}$$

In context, we want to control the number of partition classes from $\mathscr{P}^{(j)}$ contained in $\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})$ for a fixed polyad $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$. The following definition makes this precise.

Definition 3.1 (**family of partitions**). *For a vector* $\boldsymbol{a} = (a_1, \ldots, a_{k-1})$ *of positive integers, we say* $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}) = \{\mathscr{P}^{(1)}, \ldots, \mathscr{P}^{(k-1)}\}$ *is a* family of partitions on V *if* $\mathscr{P}^{(1)}$ *is a partition of* V *into* a_1 *classes and* $\mathscr{P}^{(j)}$ *is a partition of* $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ *refining* $\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ *where, for every* $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$, $|\{\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}\colon \mathcal{P}^{(j)} \subseteq \mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\}| = a_j$. *Moreover, we say* $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ *is* t-bounded *if* $\max\{a_1, \ldots, a_{k-1}\} \le t$.

We also want the families $\mathscr{P}$ to be 'equitable', in the following sense.

Definition 3.2 (($\varepsilon, \boldsymbol{a}$)**-equitable**). *Suppose* $\varepsilon > 0$, $\boldsymbol{a} = (a_1, \ldots, a_{k-1})$ *is a vector of positive integers and* $|V| = n$. *We say a family of partitions* $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ *on* V *is* $(\varepsilon, \boldsymbol{a})$-equitable *if* $||V_i| - |V_j|| \le 1$ *for all* $i, j \in [a_1]$ *and if for every* $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$, *the* $(k, k-1)$*-complex* $\boldsymbol{\mathcal{P}}_{\hat{\mathcal{P}}^{(k-1)}}$ *(cf. (6)) is* $(\varepsilon, (1/a_2, \ldots, 1/a_{k-1}))$*-regular. For* $\eta > 0$, *we say the* $(\varepsilon, \boldsymbol{a})$*-equitable family of partitions* $\mathscr{P}$ *is* $(\eta, \varepsilon, \boldsymbol{a})$-equitable *if, additionally,* $|[V]^k \setminus \mathrm{Cross}_k(\mathscr{P}^{(1)})| \le \eta\binom{n}{k}$.

The following definition describes when a hypergraph is 'perfectly regular' w.r.t. a family of partitions $\mathscr{P}$.

Definition 3.3 (**perfectly** ε**-regular**). *Let* $\varepsilon > 0$ *be given. Let* $\mathcal{H}^{(k)}$ *be a* k*-graph on vertex set* V *and let* $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ *be a family of partitions on* V. *We say* $\mathcal{H}^{(k)}$ *is* perfectly ε-regular *w.r.t.* $\mathscr{P}$ *if for every* $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ *we have that* $\mathcal{H}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ *is* (ε, d)*-regular w.r.t.* $\hat{\mathcal{P}}^{(k-1)}$, *for some* $d \in [0, 1]$.

The regular-approximation lemma of Rödl and Schacht is given as follows (see Theorem 14 of [11]).

Theorem 3.4 (**Regular Approximation Lemma**). *Let* $k \ge 2$ *be a fixed integer. For all positive constants* η *and* ν *and every function* $\varepsilon : \mathbb{N}^{k-1} \to (0, 1]$, *there exist integers* $t = t_{\text{Thm. 3.4}}$ *and* $n_0 = n_{\text{Thm. 3.4}}$ *so that for every* k*-uniform hypergraph* $\mathcal{G}^{(k)}$ *with* $|V(\mathcal{G}^{(k)})| = n \ge n_0$, *there exist an* $(\eta, \varepsilon(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$*-equitable and* t*-bounded family of partitions* $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ *and a* k*-uniform hypergraph* $\mathcal{H}^{(k)}$ *which is perfectly* $\varepsilon(\boldsymbol{a}^{\mathscr{P}})$*-regular w.r.t.* $\mathscr{P}$ *and where* $|\mathcal{G}^{(k)} \triangle \mathcal{H}^{(k)}| < \nu n^k$.

In Remark 3.5 below, we describe a 'k-partite' version of Theorem 3.4, which was not specifically stated in [11], but which follows[2] from the proof in [11].

Remark 3.5. In Theorem 3.4, suppose $\mathcal{G}^{(k)}$ is k-partite with k-partition $V(\mathcal{G}^{(k)}) = U_1 \cup \cdots \cup U_k$. If each $|U_i| = n_i$, $1 \le i \le k$, is sufficiently large, then the vertex partition $\mathscr{P}^{(1)} = \{V_1, \ldots, V_{a_1}\}$ of $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ can be taken to refine $U_1 \cup \cdots \cup U_k$, i.e., for each $1 \le j \le a_1$, there exists $1 \le i \le k$ so that $V_j \subseteq U_i$, and for each $1 \le i \le k$, there exist b_i and indices $1 \le j_1 < \cdots < j_{b_i} \le a_1$ so that $U_i = V_{j_1} \cup \cdots \cup V_{j_{b_i}}$. In this case, we shall rewrite $\mathscr{P}^{(1)} = \{V_1, \ldots, V_{a_1}\}$ as $\mathscr{P}^{(1)} = \{V_{ij} : 1 \le i \le k,\ 1 \le j \le b_i\}$ (so that $a_1 = b_1 + \cdots + b_k$), where for each $1 \le i \le k$, $U_i = V_{i1} \cup \cdots \cup V_{ib_i}$ and where $||V_{ij}| - |V_{i'j'}|| \le 1$ for each $i, i' \in [k]$ and $(j, j') \in [b_i] \times [b_i']$. In this context, the hypergraph $\mathcal{H}^{(k)}$ can be taken as k-partite and where $|\mathcal{G}^{(k)} \triangle \mathcal{H}^{(k)}| < \nu n_1 \cdots n_k$.

3.2. **Equitable partitions and** (d, ζ)**-uniformity.** Suppose $\mathcal{H}^{(k)}$ is perfectly ε-regular w.r.t. $(\varepsilon, \boldsymbol{a})$-equitable family of partitions $\mathscr{P}$. For a sequence of k vertex classes $\boldsymbol{V}$ from $\mathscr{P}^{(1)}$, the following proposition asserts that $\mathcal{H}^{(k)}[\boldsymbol{V}]$ is $(d_{\mathcal{H}^{(k)}}(\boldsymbol{V}), \delta)$-uniform.

Proposition 3.6. *For all* $k \ge 2$ *and* $\boldsymbol{a} = (a_1 = k, a_2, \ldots, a_{k-1})$ *and* $\delta > 0$, *there exist* $\varepsilon > 0$ *and positive integer* m_0 *so that the following holds: Suppose* $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ *is an* $(\varepsilon, \boldsymbol{a})$*-equitable family of partitions on a set* X *where* $\mathscr{P}^{(1)} = (X_1, \ldots, X_k) = \boldsymbol{X}$, *i.e.,* $X = X_1 \cup \cdots \cup X_k$, *and suppose that a (*$k$*-partite)* k*-uniform hypergraph* $\mathcal{H}^{(k)}$ *with vertex set* $X = V(\mathcal{H}^{(k)})$ *is perfectly* ε*-regular w.r.t.* $\mathscr{P}$ *where* $|X_1|, \ldots, |X_k| \ge m_0$. *Then* $\mathcal{H}^{(k)} = \mathcal{H}^{(k)}[\boldsymbol{X}]$ *is* $(d_{\mathcal{H}^{(k)}}(\boldsymbol{X}), \delta)$*-uniform.*

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[2]A well-known feature of graph (hypergraph) regularity lemmas is that, if a given graph (hypergraph) is equipped with a fixed vertex partition, then a 'regular partition' of this graph (hypergraph) can be ensured which refines the given vertex partition. For example, Gowers [5] formulated his hypergraph regularity lemma in this way.

Before we may give the proof of Proposition 3.6, we require the following observation. In the context above, fix a polyad $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ and let $\boldsymbol{W} = (W_1, \dots, W_k)$, $W_i \subseteq X_i$, $|W_i| > \delta |X_i|$, $1 \le i \le k$ be given. Since $\mathscr{P}$ is $(\varepsilon, \boldsymbol{a})$-equitable, $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}_{\hat{\mathcal{P}}^{(k-1)}}$ (cf. (6)) is $(\varepsilon, (1/a_2, \dots, 1/a_{k-1}))$-regular. We need this regularity to be preserved when $\boldsymbol{\mathcal{P}}$ is induced on $\boldsymbol{W}$.

Fact 3.7. *For all $j \ge i \ge 2$ and $d_i, \dots, d_2, \tilde{\varepsilon} > 0$, there exist $\varepsilon = \varepsilon_{\text{Fact. 3.7}} > 0$ and positive m_0 so that whenever $\boldsymbol{\mathcal{P}} = \{\mathcal{P}^{(h)}\}_{h=1}^{i}$ is an $(\varepsilon, (d_2, \dots, d_i))$-regular (j, i)-complex with j-partition $\mathcal{P}^{(1)} = X_1 \cup \dots \cup X_j$, $|X_a| \ge m_0$, $1 \le a \le j$, then for all vectors $\boldsymbol{W} = (W_1, \dots, W_j)$ of subsets $W_a \subseteq X_a$, $|W_a| > \tilde{\varepsilon}|X_a|$, $1 \le a \le j$, the (j, i)-complex $\boldsymbol{\mathcal{P}}[\boldsymbol{W}] \overset{\text{def}}{=} \{\mathcal{P}^{(h)}[\boldsymbol{W}]\}_{h=1}^{i}$ is $(\tilde{\varepsilon}, (d_2, \dots, d_i))$-regular.*

Proof of Proposition 3.6. Let $k \ge 2$ and $\boldsymbol{a} = (a_1 = k, a_2, \dots, a_{k-1})$ and $\delta > 0$ be given. Set $\gamma = \delta^4/5$ and let $\varepsilon_1 = \varepsilon_{\text{Lem. 2.4}}(k, k-1, \gamma, 1/a_{k-1}, \dots, 1/a_2) > 0$ be the constant guaranteed by Lemma 2.4 (dense counting lemma). Let $\varepsilon_2 = \varepsilon_{\text{Fact. 3.7}}(k, k-1, 1/a_{k-1}, \dots, 1/a_2, \varepsilon_1) > 0$ be the constant guaranteed by Fact 3.7. Set $\varepsilon = \min\{\gamma^k/4, \varepsilon_1, \varepsilon_2\}$ and take m_0 sufficiently large whenever needed. Let $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$, $\boldsymbol{X}$ and $\mathcal{H}^{(k)}$ be given as in Proposition 3.6 and let $\boldsymbol{W} = (W_1, \dots, W_k)$ be such that $W_i \subseteq X_i$, $|W_i| > \delta |X_i|$, $1 \le i \le k$. Observe that

$$|\mathcal{H}^{(k)}| = \sum_{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}} \big|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big| = \sum_{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}} d(\mathcal{H}^{(k)}|\hat{\mathcal{P}}^{(k-1)})\big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big|, \quad \text{and}$$

$$|\mathcal{H}^{(k)}[\boldsymbol{W}]| = \sum_{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}} \big|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}[\boldsymbol{W}])\big| = \sum_{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}} d(\mathcal{H}^{(k)}|\hat{\mathcal{P}}^{(k-1)}[\boldsymbol{W}])\big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}[\boldsymbol{W}])\big|.$$

For a fixed $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$, Fact 3.7 gives that $\boldsymbol{\mathcal{P}}_{\hat{\mathcal{P}}^{(k-1)}}[\boldsymbol{W}]$ is an $(\varepsilon_1, (1/a_2, \dots, 1/a_{k-1}))$-regular $(k, k-1)$-complex. Lemma 2.4 therefore implies

$$\big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}[\boldsymbol{W}])\big| = (1 \pm \gamma) \prod_{i=2}^{k-1} \left(\tfrac{1}{a_i}\right)^{\binom{k}{i}} \times |W_1| \dots |W_k| \ge \tfrac{1}{2}\delta^k \prod_{i=2}^{k-1} \left(\tfrac{1}{a_i}\right)^{\binom{k}{i}} \times |X_1| \dots |X_k|, \quad \text{and}$$

$$\big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big| = (1 \pm \gamma) \prod_{i=2}^{k-1} \left(\tfrac{1}{a_i}\right)^{\binom{k}{i}} \times |X_1| \dots |X_k| \le 2 \prod_{i=2}^{k-1} \left(\tfrac{1}{a_i}\right)^{\binom{k}{i}} \times |X_1| \dots |X_k|. \quad (7)$$

Then $|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}[\boldsymbol{W}])| > \varepsilon |\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|$, implying $d(\mathcal{H}^{(k)}|\hat{\mathcal{P}}^{(k-1)}[\boldsymbol{W}]) = d(\mathcal{H}^{(k)}|\hat{\mathcal{P}}^{(k-1)}) \pm \varepsilon$, and so

$$|\mathcal{H}^{(k)}[\boldsymbol{W}]| = \sum_{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}} d(\mathcal{H}^{(k)}|\hat{\mathcal{P}}^{(k-1)})\big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}[\boldsymbol{W}])\big| \pm \varepsilon \sum_{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}} \big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}[\boldsymbol{W}])\big|$$

$$\overset{(7)}{\Longrightarrow} \quad d_{\mathcal{H}^{(k)}}(\boldsymbol{W}) = \pm\varepsilon + (1 \pm \gamma) \prod_{i=2}^{k-1} \left(\tfrac{1}{a_i}\right)^{\binom{k}{i}} \times \sum_{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}} d(\mathcal{H}^{(k)}|\hat{\mathcal{P}}^{(k-1)}).$$

On the other hand, (7) also implies

$$d_{\mathcal{H}^{(k)}}(\boldsymbol{X}) = (1 \pm \gamma) \prod_{i=2}^{k-1} \left(\tfrac{1}{a_i}\right)^{\binom{k}{i}} \times \sum_{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}} d(\mathcal{H}^{(k)}|\hat{\mathcal{P}}^{(k-1)}) \quad \Longrightarrow$$

$$d_{\mathcal{H}^{(k)}}(\boldsymbol{X}) - \gamma^{1/2} - \varepsilon \le \frac{1-\gamma}{1+\gamma} d_{\mathcal{H}^{(k)}}(\boldsymbol{X}) - \varepsilon \le d_{\mathcal{H}^{(k)}}(\boldsymbol{W}) \le \frac{1+\gamma}{1-\gamma} d_{\mathcal{H}^{(k)}}(\boldsymbol{X}) + \varepsilon \le d_{\mathcal{H}^{(k)}}(\boldsymbol{X}) + \gamma^{1/2} + \varepsilon,$$

from which $d_{\mathcal{H}^{(k)}}(\boldsymbol{W}) = d_{\mathcal{H}^{(k)}}(\boldsymbol{X}) \pm \delta$ follows. □

Proof of Fact 3.7. It suffices to prove the statement for $j = i$, on which we induct and where the base case $i = 2$ is well-known (see Fact 1.5, in [8]). Now, let $i \ge 3$ and $d_i, \dots, d_2, \tilde{\varepsilon} > 0$ be given. Let $\varepsilon_1 = \varepsilon_{\text{Lem. 2.4}}(i, i-1, 1/2, d_{i-1}, \dots, d_2) > 0$ be the constant guaranteed by Lemma 2.4 and $\varepsilon_2 = \varepsilon_{\text{Fact 3.7}}(i, i-1, d_{i-1}, \dots, d_2, \tilde{\varepsilon}) > 0$ be the constant guaranteed by our induction hypothesis. Take $\varepsilon = \frac{1}{3}\min\{\tilde{\varepsilon}^{i+1}, \varepsilon_1, \varepsilon_2\}$ and m_0 sufficiently large. With these constants, let $\boldsymbol{\mathcal{P}} = \{\mathcal{P}^{(h)}\}_{h=1}^{i}$ and vector $\boldsymbol{W}$ of subsets be given as in the hypothesis of Fact 3.7. Let $\mathcal{Q}^{(i-1)} \subseteq \mathcal{P}^{(i-1)}[\boldsymbol{W}]$ satisfy $|\mathcal{K}_i(\mathcal{Q}^{(i-1)})| >$

$\tilde{\varepsilon}|\mathcal{K}_i(\mathcal{P}^{(i-1)}[\boldsymbol{W}])|$. Since the $(i,i-1)$-complex $\{\mathcal{P}^{(h)}[\boldsymbol{W}]\}_{h=1}^{i-1}$ is $(\tilde{\varepsilon},(d_2,\dots,d_{i-1}))$-regular (by induction), Lemma 2.4 implies

$$|\mathcal{K}_i(\mathcal{P}^{(i-1)}[\boldsymbol{W}])| > \frac{1}{2}\prod_{h=2}^{i-1} d_h^{\binom{i}{h}} \times \prod_{a=1}^{i}|W_a| \quad \text{and} \quad |\mathcal{K}_i(\mathcal{P}^{(i-1)})| < \frac{3}{2}\prod_{h=2}^{i-1} d_h^{\binom{i}{h}} \prod_{a=1}^{i}|X_a|.$$

As such, $|\mathcal{K}_i(\mathcal{Q}^{(i-1)})| > \varepsilon|\mathcal{K}_i(\mathcal{P}^{(i-1)})|$ and so the (ε,d_i)-regularity of $\mathcal{P}^{(i)}$ w.r.t. $\mathcal{P}^{(i-1)}$ implies $|\mathcal{P}^{(i)}[\boldsymbol{W}]\cap\mathcal{K}_i(\mathcal{Q}^{(i-1)})| = |\mathcal{P}^{(i)}\cap\mathcal{K}_i(\mathcal{Q}^{(i-1)})| = (d_i\pm\varepsilon)|\mathcal{K}_i(\mathcal{Q}^{(i-1)})|$. □

4. Proof of Theorem 1.3

Let an integer $k\ge 2$ and a constant $\zeta_0>0$ be given. Without loss of generality, assume $\zeta_0<0.01$ and also assume $k\ge 3$, since the case $k=2$ is, in fact, proven in Section 2.1. Our definitions of the promised constants $\zeta=\zeta_{\text{Thm. 1.3}}$, $c=c_{\text{Thm. 1.3}}$ and $s_0=s_{\text{Thm. 1.3}}$ depend on auxiliary parameters which we now define. For positive integer variables $a_2,\dots,a_{k-1}$, let $\varepsilon'(a_2,\dots,a_{k-1})=\varepsilon_{\text{Prop. 3.6}}(k,a_2,\dots,a_{k-1},\delta=\zeta_0^{2k})>0$ be the function guaranteed by Proposition 3.6. Let

$$\begin{aligned}
\varepsilon(a_1,a_2,\dots,a_{k-1}) &= \varepsilon_{\text{Thm. 1.5}}(k,d_2=1/a_2,\dots,d_{k-1}=1/a_{k-1},\varepsilon'(a_2,\dots,a_{k-1}))\,,\\
c_{\text{Thm. 1.5}}(a_2,\dots,a_{k-1}) &= c_{\text{Thm. 1.5}}(k,d_2=1/a_2,\dots,d_{k-1}=1/a_{k-1},\varepsilon'(a_2,\dots,a_{k-1}))\,,\\
s_{\text{Thm. 1.5}}(a_2,\dots,a_{k-1}) &= s_{\text{Thm. 1.5}}(k,d_2=1/a_2,\dots,d_{k-1}=1/a_{k-1},\varepsilon'(a_2,\dots,a_{k-1}))
\end{aligned} \tag{8}$$

be the functions guaranteed by Theorem 1.5 (ε is constant in the variable a_1). Let $t=t_{\text{Thm. 3.4}}(\nu=\zeta_0^{4k},\varepsilon(a_1,a_2,\dots,a_{k-1}))$ be the constant guaranteed by Theorem 3.4. Set

$$s_{\text{Thm. 1.5}} = \max s_{\text{Thm. 1.5}}(a_2,\dots,a_{k-1}) \quad \text{and} \quad c_{\text{Thm. 1.5}} = \min\{c_{\text{Thm. 1.5}}(a_2,\dots,a_{k-1}),1\}\,,$$

where the max and min above are both taken over $1\le a_2,\dots,a_{k-1}\le t$. Define

$$\zeta=\frac{1}{2t}\,,\quad s_0=\frac{3\cdot 2^{k+3}\ln(2t)}{\zeta_0^{32k}c_{\text{Thm. 1.5}}} \quad \text{and} \quad c=\frac{\zeta_0^{32k}c_{\text{Thm. 1.5}}}{24t}\,. \tag{9}$$

Now, with ζ given in (9), let $\mathcal{G}^{(k)}$ be a (ρ,ζ)-uniform (k,k)-cylinder with k-partition $V(\mathcal{G}^{(k)})=U_1\cup\dots\cup U_k$, where $\rho\in[0,1]$, and each $|U_i|=n_i$, $1\le i\le k$, is sufficiently large. For fixed $s_0\le s\le n_1$, we show that all but $\exp\{-cs\}\binom{n_1}{s}$ sets $S\subseteq\binom{U_1}{s}$ yield $\boldsymbol{S}=(S,U_2,\dots,U_k)$ for which $\mathcal{G}^{(k)}[\boldsymbol{S}]$ is (ρ,ζ_0)-uniform.

With $\nu=\zeta_0^{4k}$ and function $\varepsilon(a_1,a_2,\dots,a_{k-1})$ of (8), apply Theorem 3.4 (see Remark 3.5) to $\mathcal{G}^{(k)}$ to obtain k-uniform hypergraph $\mathcal{H}^{(k)}$ and $(\varepsilon(\boldsymbol{a}^{\mathscr{P}}),\boldsymbol{a}^{\mathscr{P}})$-equitable and t-bounded family of partitions $\mathscr{P}=\mathscr{P}(k-1,\boldsymbol{a}^{\mathscr{P}})$ with respect to which $\mathcal{H}^{(k)}$ is perfectly $\varepsilon(\boldsymbol{a}^{\mathscr{P}})$-regular and for which $|\mathcal{G}^{(k)}\triangle\mathcal{H}^{(k)}|<\nu n_1\cdots n_k$. From this application, $a_1=b_1+\dots+b_k$ (recall the notation of Remark 3.5), $a_2,\dots,a_{k-1}$ are now fixed, as are $\varepsilon'(a_2,\dots,a_{k-1})$ and $\varepsilon(a_1,\dots,a_{k-1})$ from (8), which we now abbreviate to ε' and ε, resp. Note that, after this application of Theorem 3.4, the constants above relate as follows:

$$\tfrac{1}{k},\zeta_0 \gg \nu=\zeta_0^{4k}\ge\min\left\{\nu,\tfrac{1}{a_2},\dots,\tfrac{1}{a_{k-1}}\right\}\gg\varepsilon'\gg\varepsilon\ge\min\begin{cases}\varepsilon\\ 2\zeta=\tfrac{1}{t}\gg\max\left\{\tfrac{1}{s_0},c\right\}.\end{cases} \tag{10}$$

We now consider some notation (see Remark 3.5). Fix $\boldsymbol{j}=(j_1,\dots,j_k)\in[b_1]\times\dots\times[b_k]\stackrel{\text{def}}{=}\mathbb{J}$ and write $\boldsymbol{V_j}=(V_{1j_1},\dots,V_{kj_k})$. Call $\boldsymbol{j}$ a *typical vector* if $|(\mathcal{G}^{(k)}\triangle\mathcal{H}^{(k)})[\boldsymbol{V_j}]|<\nu^{1/2}|V_{1j_1}|\dots|V_{kj_k}|$, and write $\mathbb{J}_{\text{typ}}\subseteq\mathbb{J}$ for the set of typical vectors. Clearly,

$$\left|\mathbb{J}_{\text{typ}}\right|\ge\left(1-2\nu^{1/2}\right)|\mathbb{J}|=\left(1-2\nu^{1/2}\right)b_1\cdots b_k\,, \tag{11}$$

since otherwise, $|\mathcal{G}^{(k)}\triangle\mathcal{H}^{(k)}|\ge 2\nu b_1\cdots b_k\cdot\lfloor n_1/b_1\rfloor\cdots\lfloor n_k/b_k\rfloor>\nu n_1\cdots n_k$. For a subset $S\subseteq U_1$, write $S_{j_1}=S\cap V_{1j_1}$ and $\boldsymbol{S_j}=(S_{j_1},V_{2j_2},\dots,V_{kj_k})$. More generally, for a vector $\boldsymbol{W}=(W_1,\dots,W_k)$ of subsets $W_1\subseteq U_1,\dots,W_k\subseteq U_k$, write $\boldsymbol{W_j}=(W_{1j_1},\dots,W_{kj_k})$, where $W_{ij_i}=W_i\cap V_{ij_i}$ for $1\le i\le k$. Call $S\in\binom{U_1}{s}$ a *typical set* if:

(1) $|(\mathcal{G}^{(k)}\triangle\mathcal{H}^{(k)})[\boldsymbol{S}]|<2\nu s\cdot n_2\cdots n_k$, and for each $j\in[b_1]$, $s/(2b_1)\le|S_j|\le 2s/b_1$;
(2) for each $\boldsymbol{j}\in\mathbb{J}_{\text{typ}}$, $\mathcal{H}^{(k)}[\boldsymbol{S_j}]$ is $(d_{\mathcal{H}^{(k)}}(\boldsymbol{S_j}),\zeta_0^{2k})$-uniform with $d_{\mathcal{H}^{(k)}}(\boldsymbol{S_j})=\rho\pm 3\zeta_0^{2k}$.

Theorem 1.3 is established by the following claim.

Claim 4.1. *For each typical set* $S \in \binom{U_1}{s}$, $\mathcal{G}^{(k)}[\boldsymbol{S}]$ *is* (ρ, ζ_0)*-uniform. Moreover, all but* $\exp\{-cs\}\binom{n_1}{s}$ *many* $S \in \binom{U_1}{s}$ *are typical sets.*

Proof of Claim 4.1 (first assertion). Fix a typical set $S \in \binom{U_1}{s}$, and then fix $\boldsymbol{W} = (W_1, \ldots, W_k)$, where $W_1 \subseteq S, W_2 \subseteq U_2, \ldots, W_k \subseteq U_k$, and

$$|W_1| > \zeta_0 s\,,\ |W_2| > \zeta_0 n_2\,,\ \ldots\,,\ |W_k| > \zeta_0 n_k\,. \tag{12}$$

To show that $d_{\mathcal{G}^{(k)}}(\boldsymbol{W}) = \rho \pm \zeta_0$, it is enough to show

$$d_{\mathcal{H}^{(k)}}(\boldsymbol{W}) = \rho \pm \zeta_0^2\,. \tag{13}$$

Indeed, $\mathcal{G}^{(k)}[\boldsymbol{W}]$ satisfies $|\mathcal{G}^{(k)}[\boldsymbol{W}]| = |\mathcal{H}^{(k)}[\boldsymbol{W}]| \pm |(\mathcal{G}^{(k)} \triangle \mathcal{H}^{(k)})[\boldsymbol{W}]|$, where Condition (1) ensures $|(\mathcal{G}^{(k)} \triangle \mathcal{H}^{(k)})[\boldsymbol{W}]| \leq |(\mathcal{G}^{(k)} \triangle \mathcal{H}^{(k)})[\boldsymbol{S}]| \leq 2\nu s \cdot n_2 \cdots n_k \overset{(10)}{=} 2\zeta_0^{4k} s \cdot n_2 \cdots n_k$. As such, $d_{\mathcal{G}^{(k)}}(\boldsymbol{W}) = \rho \pm \zeta_0^2 \pm 2\zeta_0^{4k}\zeta_0^{-k} = \rho \pm \zeta_0$, where we used (12). To establish (13), let $\mathcal{F}^{(k)} \in \{\mathcal{H}^{(k)}, \mathcal{K}^{(k)} = K^{(k)}[U_1, \ldots, U_k]\}$ and observe that $|\mathcal{F}^{(k)}[\boldsymbol{W}]| = \sum_{\boldsymbol{j} \in \mathbb{J}} |\mathcal{F}^{(k)}[\boldsymbol{W_j}]|$. Then $\sum_{\boldsymbol{j} \in \mathbb{J}_{\mathrm{typ}}} |\mathcal{F}^{(k)}[\boldsymbol{W_j}]| \leq |\mathcal{F}^{(k)}[\boldsymbol{W}]| \leq 8\nu^{1/2} s \cdot n_2 \cdots n_k + \sum_{\boldsymbol{j} \in \mathbb{J}_{\mathrm{typ}}} |\mathcal{F}^{(k)}[\boldsymbol{W_j}]|$ follows from (11) and Condition (1), since any $\boldsymbol{j} = (j_1, \ldots, j_k) \in \mathbb{J} \setminus \mathbb{J}_{\mathrm{typ}}$ satisfies $|W_{1j_1}| \leq |S_{j_1}| \leq 2s/b_1$ and $|W_{ij_i}| \leq |V_{ij_i}| \leq \lceil n_i/b_i \rceil$, $2 \leq i \leq k$. Now, call $\boldsymbol{j} = (j_1, \ldots, j_k) \in \mathbb{J}_{\mathrm{typ}}$ *big* if $|W_{1j_1}| > \zeta_0^{2k}|S_{j_1}|$, $|W_{2j_2}| > \zeta_0^{2k}|V_{2j_2}|, \ldots, |W_{kj_k}| > \zeta_0^{2k}|V_{kj_k}|$, and write $\mathbb{J}_{\mathrm{typ}}^{\mathrm{big}} \subseteq \mathbb{J}_{\mathrm{typ}}$ for the set of all big and typical vectors. Then $\sum_{\boldsymbol{j} \in \mathbb{J}_{\mathrm{typ}}^{\mathrm{big}}} |\mathcal{F}^{(k)}[\boldsymbol{W_j}]| \leq \sum_{\boldsymbol{j} \in \mathbb{J}_{\mathrm{typ}}} |\mathcal{F}^{(k)}[\boldsymbol{W_j}]| \leq 4\zeta_0^{2k} s n_2 \cdots n_k + \sum_{\boldsymbol{j} \in \mathbb{J}_{\mathrm{typ}}^{\mathrm{big}}} |\mathcal{F}^{(k)}[\boldsymbol{W_j}]|$, since each $\boldsymbol{j} = (j_1, \ldots, j_k) \in \mathbb{J}_{\mathrm{typ}} \setminus \mathbb{J}_{\mathrm{typ}}^{\mathrm{big}}$ satisfies $|W_{1j_1}| < \zeta_0^{2k}|S_{j_1}| \leq 2\zeta_0^{2k} s/b_1$ (see Condition (1)) or, for some $2 \leq i \leq k$, $|W_{ij_i}| < \zeta_0^{2k}|V_{ij_i}| \leq \zeta_0^{2k}\lceil n_i/b_i \rceil$. Using $\nu = \zeta_0^{4k}$ in (10) we have, altogether,

$$\left|\mathcal{F}^{(k)}[\boldsymbol{W}]\right| = \pm 12\zeta_0^{2k} s \cdot n_2 \cdots n_k + \sum_{\boldsymbol{j} \in \mathbb{J}_{\mathrm{typ}}^{\mathrm{big}}} \left|\mathcal{F}^{(k)}[\boldsymbol{W_j}]\right|. \tag{14}$$

Now, fix $\boldsymbol{j} \in \mathbb{J}_{\mathrm{typ}}^{\mathrm{big}}$ and let $\mathcal{F}^{(k)} = \mathcal{H}^{(k)}$. Condition (2) implies that $\mathcal{H}^{(k)}[\boldsymbol{S_j}]$ is $(\rho, 4\zeta_0^{2k})$-uniform, which with $\boldsymbol{j}$ being big, $|\mathcal{H}^{(k)}[\boldsymbol{W_j}]| = (\rho \pm 4\zeta_0^{2k})|W_{1j_1}| \cdots |W_{kj_k}|$. Then (14) yields $|\mathcal{H}^{(k)}[\boldsymbol{W}]| = \pm 12\zeta_0^{2k} s \cdot n_2 \cdots n_k + (\rho \pm 4\zeta_0^{2k}) \sum_{\boldsymbol{j} \in \mathbb{J}_{\mathrm{typ}}^{\mathrm{big}}} |W_{1j_1}| \cdots |W_{kj_k}|$ and so

$$\left|\mathcal{H}^{(k)}[\boldsymbol{W}]\right| = \pm 12\zeta_0^{2k} s \cdot n_2 \cdots n_k + \left(\rho \pm 4\zeta_0^{2k}\right)\Big[\left|\mathcal{K}^{(k)}[\boldsymbol{W}]\right| \pm 12\zeta_0^{2k} s \cdot n_2 \cdots n_k\Big],$$

which implies $d_{\mathcal{H}^{(k)}}(\boldsymbol{W}) = \rho \pm 4\zeta_0^{2k} \pm 24\zeta_0^{k} = \rho \pm \zeta_0^2$, where we used (12) (and $\zeta_0 < 0.01$ and $k \geq 3$). □

Proof of Claim 4.1 (part 2). Using the definition, we enumerate the 'atypical' sets. For the first part of Condition (1), apply Fact 1.6 with $\eta = \nu$ to the hypergraph $\mathcal{D}^{(k)} = \mathcal{G}^{(k)} \triangle \mathcal{H}^{(k)}$ (of density $d_{\mathcal{D}^{(k)}}(\boldsymbol{U}) < \nu$) to conclude all but $\exp\{-\nu^8 s/6\}\binom{n_1}{s} = \exp\{-\zeta_0^{32k} s/6\}\binom{n_1}{s}$ sets $S \in \binom{U_1}{s}$ satisfy $d_{\mathcal{D}^{(k)}}(\boldsymbol{S}) = d_{\mathcal{D}^{(k)}}(\boldsymbol{U}) \pm \nu < 2\nu$, so that $|\mathcal{G}^{(k)} \triangle \mathcal{H}^{(k)}[\boldsymbol{S}]| < 2\nu s \cdot n_2 \cdots n_k$. For the second part of Condition (1), fix $j \in [b_1]$ and recall $\lfloor n_1/b_1 \rfloor \leq |V_{1j}| \leq \lceil n_1/b_1 \rceil$. By the Chernoff-Hoeffding inequality (1), all but $2\exp\{-s/(12b_1)\}\binom{n_1}{s} \leq 2\exp\{-s/(12t)\}\binom{n_1}{s}$ sets $S \in \binom{U_1}{s}$ satisfy $s/(2b_1) \leq |S \cap V_{1j}| \leq 2s/b_1$. Over all $j \in [b_1]$, all but $2b_1 \exp\{-s/(12t)\}\binom{n_1}{s} \leq 2t\exp\{-s/(12t)\}\binom{n_1}{s}$ satisfy this property.

To argue the density assertion of Condition (2), fix $\boldsymbol{j} = (j_1, \ldots, j_k) \in \mathbb{J}_{\mathrm{typ}}$. Observe that $d_{\mathcal{G}^{(k)}}(\boldsymbol{V_j}) = \rho \pm \zeta$ since $\mathcal{G}^{(k)}$ is (ρ, ζ)-uniform and each $V_{ij_i} \subset V_i$, $1 \leq i \leq k$, satisfies $|V_{ij_i}| \geq \lfloor n_i/b_i \rfloor \geq n_i/(2t) = \zeta n_i$ (cf. (10)). Now, since $\boldsymbol{j} \in \mathbb{J}_{\mathrm{typ}}$, $d_{\mathcal{H}^{(k)}}(\boldsymbol{V_j}) = \rho \pm \zeta \pm \nu^{1/2} = \rho \pm 2\zeta_0^{2k}$. Apply Fact 1.6 with an arbitrary integer $s/(2b_1) \leq s_{j_1} \leq 2s/b_1$ (where $s/2b_1 \geq s/2t$ is 'large enough' (cf. (9))) so that all but $\exp\{-\zeta_0^{16k} s_{j_1}/6\}\binom{|V_{1j_1}|}{s_{j_1}} \leq \exp\{-\zeta_0^{16k} s/(12b_1)\}\binom{|V_{1j_1}|}{s_{j_1}} \leq \exp\{-\zeta_0^{16k} s/(12t)\}\binom{|V_{1j_1}|}{s_{j_1}}$ sets $S_{j_1} \in \binom{V_{1j_1}}{s_{j_1}}$ satisfy $d_{\mathcal{H}^{(k)}}(\boldsymbol{S_j}) = d_{\mathcal{H}^{(k)}}(\boldsymbol{V_j}) \pm \zeta_0^{2k} = \rho \pm 3\zeta_0^{2k}$. This implies that all but

$$\exp\Big\{-\tfrac{\zeta_0^{16k}}{12t}s\Big\} \sum_{s/(2b_1) \leq s_{j_1} \leq 2s/b_1} \tbinom{|V_{1j_1}|}{s_{j_1}}\tbinom{n_1 - |V_{1j_1}|}{s - s_{j_1}} \leq \exp\Big\{-\tfrac{\zeta_0^{16k}}{12t}s\Big\}\tbinom{n_1}{s}$$

sets $S \in \binom{U_1}{s}$ satisfy that $s/(2b_1) \le |S_{j_1}| \le 2s/b_1$ and that $d_{\mathcal{H}^{(k)}}(\boldsymbol{S_j}) = \rho \pm 3\zeta_0^{2k}$. Over all $\boldsymbol{j} \in \mathbb{J}_{\rm typ}$, we have all but $b_1 \cdots b_k \exp\{-\zeta_0^{16k} s/(12t)\}\binom{n_1}{s} \le t^k \exp\{-\zeta_0^{16k} s/(12t)\}\binom{n_1}{s}$ such sets S.

We now argue the uniformity assertion of Condition (2), and in fact, we argue a stronger property. To that end, fix $\boldsymbol{j} \in \mathbb{J}_{\rm typ}$ and write $\mathscr{P}_{\boldsymbol{j}}$ for the subfamily of $\mathscr{P}$ induced on the vertex partition $\boldsymbol{V_j}$ and $\hat{\mathscr{P}}_{\boldsymbol{j}}^{(k-1)}$ for its corresponding family of polyads. For $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\boldsymbol{j}}^{(k-1)}$, write $\mathcal{H}^{(k)}_{\hat{\mathcal{P}}^{(k-1)}} = \mathcal{H}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$, write $\boldsymbol{\mathcal{H}}_{\hat{\mathcal{P}}^{(k-1)}}$ for the (k,k)-complex consisting of $\boldsymbol{\mathcal{P}}_{\hat{\mathcal{P}}^{(k-1)}}$ (cf. (6)) together with $\mathcal{H}^{(k)}_{\hat{\mathcal{P}}^{(k-1)}}$ and write $\boldsymbol{d}_{\hat{\mathcal{P}}^{(k-1)}} = (d_2, \ldots, d_{k-1}, d(\mathcal{H}^{(k)}|\hat{\mathcal{P}}^{(k-1)}))$. Theorem 3.4 guarantees that $\boldsymbol{\mathcal{H}}_{\hat{\mathcal{P}}^{(k-1)}}[\boldsymbol{V_j}]$ is an $(\varepsilon, \boldsymbol{d}_{\hat{\mathcal{P}}^{(k-1)}})$-regular (k,k)-complex (cf. (8)). For an integer $s/(2b_1) \le s_{j_1} \le 2s/b_1$, Theorem 1.5 guarantees that all but $\exp\{-c_{\rm Thm.\ 1.5} s_{j_1}\}\binom{|V_{1j_1}|}{s_{j_1}} \le \exp\{-c_{\rm Thm.\ 1.5} s/(2b_1)\}\binom{|V_{1j_1}|}{s_{j_1}} \le \exp\{-c_{\rm Thm.\ 1.5} s/(2t)\}\binom{|V_{1j_1}|}{s_{j_1}}$ sets $S_{j_1} \in \binom{V_{1j_1}}{s_{j_1}}$ satisfy that $\boldsymbol{\mathcal{H}}_{\hat{\mathcal{P}}^{(k-1)}}[\boldsymbol{S_j}]$ is an $(\varepsilon', \boldsymbol{d}_{\hat{\mathcal{P}}^{(k-1)}})$-regular (k,k)-complex. This implies that all but

$$\exp\Big\{-\tfrac{c_{\rm Thm.\ 1.5}}{2t} s\Big\} \sum_{s/(2b_1) \le s_{j_1} \le 2s/b_1} \binom{|V_{1j_1}|}{s_{j_1}}\binom{n_1 - |V_{1j_1}|}{s - s_{j_1}} \le \exp\Big\{-\tfrac{c_{\rm Thm.\ 1.5}}{2t} s\Big\}\binom{n_1}{s}$$

sets $S \in \binom{U_1}{s}$ satisfy that $s/(2b_1) \le |S_{j_1}| \le 2s/b_1$ and that $\boldsymbol{\mathcal{H}}_{\hat{\mathcal{P}}^{(k-1)}}[\boldsymbol{S_j}]$ is an $(\varepsilon', \boldsymbol{d}_{\hat{\mathcal{P}}^{(k-1)}})$-regular (k,k)-complex. Over all

$$\big|\hat{\mathscr{P}}_{\boldsymbol{j}}^{(k-1)}\big| = a_2^{\binom{k}{2}} \times a_3^{\binom{k}{3}} \times \cdots \times a_{k-1}^{\binom{k}{k-1}} \le t^{2^k - k} \quad \text{and} \quad \big|\mathbb{J}_{\rm typ}\big| \le t^k$$

polyads $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\boldsymbol{j}}^{(k-1)}$ and $\boldsymbol{j} \in \mathbb{J}_{\rm typ}$, all but $t^{2^k} \exp\{-c_{\rm Thm.\ 1.5} s/(2t)\}\binom{n_1}{s}$ sets $S \in \binom{U_1}{s}$ satisfy that, for each $\boldsymbol{j} \in \mathbb{J}_{\rm typ}$ and $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\boldsymbol{j}}^{(k-1)}$, $s/(2b_1) \le |S_{j_1}| \le 2s/b_1$ and that $\boldsymbol{\mathcal{H}}_{\hat{\mathcal{P}}^{(k-1)}}[\boldsymbol{S_j}]$ is a $(\varepsilon', \boldsymbol{d}_{\hat{\mathcal{P}}^{(k-1)}})$-regular (k,k)-complex. But now, fix such a set $S \in \binom{U_1}{s}$ and fix $\boldsymbol{j} \in \mathbb{J}_{\rm typ}$. Consider the family $\mathscr{P}_{\boldsymbol{j}}[\boldsymbol{S_j}]$ obtained by restricting $\mathscr{P}_{\boldsymbol{j}}$ to the vertex sets $\boldsymbol{S_j} = (S_{j_1}, V_{2j_2}, \ldots, V_{kj_k})$, i.e., for each $2 \le i \le k-1$, replace the (i,i)-cylinder $\mathcal{P}^{(i)} \in \mathscr{P}_{\boldsymbol{j}}$ with $\mathcal{P}^{(i)}[\boldsymbol{S_j}]$. By our choice of S, $\mathscr{P}_{\boldsymbol{j}}[\boldsymbol{S_j}]$ is an $(\varepsilon', (a_1 = k, a_2, \ldots, a_{k-1}))$-equitable partition of $S_{j_1} \cup V_{2j_2} \cup \cdots \cup V_{kj_k}$ with respect to which $\mathcal{H}^{(k)}[\boldsymbol{S_j}]$ is perfectly ε'-regular. Proposition 3.6 then guarantees that $\mathcal{H}^{(k)}[\boldsymbol{S_j}]$ is $(d_{\mathcal{H}^{(k)}}(\boldsymbol{S_j}), \zeta_0^{2k})$-uniform.

Combining all estimates above, the number of atypical sets $S \in \binom{U_1}{s}$ is at most

$$\Big(\exp\Big\{-\tfrac{\zeta_0^{32k}}{6} s\Big\} + 2t\exp\Big\{-\tfrac{1}{12t} s\Big\} + t^k \exp\Big\{-\tfrac{\zeta_0^{16k}}{12t} s\Big\} + t^{2^k} \exp\Big\{-\tfrac{c_{\rm Thm.\ 1.5}}{2t} s\Big\}\Big)\binom{n_1}{s}$$
$$\le 4t^{2^k} \exp\Big\{-\frac{\zeta_0^{32k} c_{\rm Thm.\ 1.5}}{12t} s\Big\}\binom{n_1}{s} \overset{(9)}{\le} \exp\Big\{-\frac{\zeta_0^{32k} c_{\rm Thm.\ 1.5}}{24t} s\Big\}\binom{n_1}{s} \overset{(9)}{=} \exp\{-cs\}\binom{n_1}{s}.$$

□

5. Appendix

To prove that Theorem 2.1 implies Theorem 1.5, we use the standard fact below (Proposition 5.1) with the following complementary parts: in an appropriate setting, (1), a 'regular' hypergraph can be split into edge-disjoint and 'regular' subhypergraphs, and (2), the union of edge-disjoint and 'regular' hypergraphs is itself 'regular'.

Proposition 5.1. *Let $\mathcal{H}^{(k-1)}$ be a $(k, k-1)$-cylinder, where $V(\mathcal{H}^{(k-1)}) = V_1 \cup \cdots \cup V_k$, $|V_i| = m_i$, $1 \le i \le k$. The following statements hold:*

1. *if $\mathcal{F}^{(k)} \subseteq \mathcal{K}_k(\mathcal{H}^{(k-1)})$ is (δ, σ)-regular w.r.t. $\mathcal{H}^{(k-1)}$, where $0 < 2\delta, \rho \le 1/2 \le \sigma \le 1$, where each $m_i \ge m_0 = m_0(k, \delta)$ is sufficiently large, and where $|\mathcal{K}_k(\mathcal{H}^{(k-1)})| \ge (m_1 \cdots m_k)/\ln(m_1 \cdots m_k)$, then there exists a partition $\mathcal{F}^{(k)} = \mathcal{F}_0^{(k)} \cup \mathcal{F}_1^{(k)} \cup \cdots \cup \mathcal{F}_p^{(k)}$, $p = \lfloor \sigma/\rho \rfloor$, where each $\mathcal{F}_i^{(k)}$, $1 \le i \le p$, is $(3\delta, \rho)$-regular w.r.t. $\mathcal{H}^{(k-1)}$, and where $\mathcal{F}_0^{(k)}$ is $(3\delta, \sigma - p\rho)$-regular w.r.t. $\mathcal{H}^{(k-1)}$;*
2. *if $\mathcal{G}_1^{(k)}, \ldots, \mathcal{G}_q^{(k)} \subseteq \mathcal{K}_k(\mathcal{H}^{(k-1)})$ are pairwise disjoint, where each $\mathcal{G}_i^{(k)}$, $1 \le i \le q$, is (γ, d_i)-regular w.r.t. $\mathcal{H}^{(k-1)}$, then $\mathcal{G}^{(k)} = \bigcup_{i=1}^q \mathcal{G}_i^{(k)}$ is $(q\gamma, d)$-regular w.r.t. $\mathcal{H}^{(k-1)}$, where $d = \sum_{i=1}^q d_i$.*

Statement (1) of Proposition 5.1 follows by a standard probabilistic argument using the Chernoff inequality, and Statement (2) follows by a standard argument using the definition of (γ, d_i)-regularity. These statements essentially appeared as Lemma 30 and Proposition 50 in [10], and as Propositions 20 and 22 in [11]. We omit their proofs.

Proof that Theorem 2.1 $\Longrightarrow$ Theorem 1.5. Let integer $k \geq 2$ and constants $d_2, \dots, d_{k-1}, \varepsilon' > 0$ be given. We define the promised constants $\varepsilon_{\text{Thm. 1.5}}$, $c_{\text{Thm. 1.5}}$ and $s_{\text{Thm. 1.5}}$ in terms of auxiliary constants. To that end, let $\varepsilon_{\text{Lem. 2.4}} = \varepsilon_{\text{Lem. 2.4}}(k, k-1, 1/2, d_{k-1}, \dots, d_2)$ be the constant guaranteed by Lemma 2.4. Define auxiliary constants

$$\rho = \min\left\{\tfrac{1}{2}\varepsilon_{\text{Lem. 2.4}}\,,\, \tfrac{1}{8}(\varepsilon')^2 \prod_{2\leq i\leq k-1} d_i^{\binom{k}{i}}\right\} \quad \text{and} \quad \varepsilon'' = \tfrac{\rho^2\varepsilon'}{4}\,.$$

Let $\varepsilon_{\text{Thm. 2.1}} = \varepsilon_{\text{Thm. 2.1}}(k, d_2, \dots, d_{k-1}, \rho, \varepsilon'')$, $c_{\text{Thm. 2.1}} = c_{\text{Thm. 2.1}}(k, d_2, \dots, d_{k-1}, \rho, \varepsilon'')$ and $s_{\text{Thm. 2.1}} = s_{\text{Thm. 2.1}}(k, d_2, \dots, d_{k-1}, \rho, \varepsilon'')$ be the constants guaranteed by Theorem 2.1. We take

$$\varepsilon = \varepsilon_{\text{Thm. 1.5}} = \tfrac{1}{3}\min\left\{\rho\,,\varepsilon_{\text{Thm. 2.1}}\right\}, \quad c = c_{\text{Thm. 1.5}} = \tfrac{1}{2}\min\left\{c_{\text{Thm. 2.1}}\,, \tfrac{\rho^8}{6}\right\},$$
$$s_0 = s_{\text{Thm. 1.5}} = \max\left\{s_{\text{Thm. 2.1}}\,, \tfrac{24}{\rho^{10}c}\right\},$$

and we take $m_0 = m_{\text{Thm. 1.5}}$ sufficiently large whenever needed.

With the constants $d_2, \dots, d_{k-1}, \varepsilon > 0$ and $m_1, \dots, m_k > m_0$ above, let $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ be an $(\varepsilon, (d_2, \dots, d_{k-1}, d_k))$-regular (k,k)-complex, as in Theorem 1.5, where $d_k \in [0,1]$ is now given. Fix integer $s_0 \leq s \leq m_1$. To prove that all but $\exp\{-c\,s\}\binom{m_1}{s}$ sets $S \in \binom{V_1}{s}$ yield $\boldsymbol{S} = (S, V_2, \dots, V_k)$ for which $\boldsymbol{\mathcal{H}}[\boldsymbol{S}]$ is an $(\varepsilon', (d_2, \dots, d_k))$-regular (k,k)-complex, we consider two cases.

Case 1 ($d_k \geq 1/2$). We first apply Statement (1) of Proposition 5.1 to the hypergraph $\mathcal{H}^{(k)} \subseteq \mathcal{K}_k(\mathcal{H}^{(k-1)})$ (where $\mathcal{F}^{(k)} = \mathcal{H}^{(k)}$, $\sigma = d_k$, $\delta = \varepsilon$). (To see that this statement applies, recall that our choice of constants were sufficient to conclude, using Lemma 2.4, that $|\mathcal{K}_k(\mathcal{H}^{(k-1)})| = \Omega(m_1 \cdots m_k)$.) Now, with the constant ρ defined above, Statement (1) of Proposition 5.1 guarantees a partition $\mathcal{H}^{(k)} = \mathcal{H}_0^{(k)} \cup \mathcal{H}_1^{(k)} \cup \dots \cup \mathcal{H}_p^{(k)}$, $p = \lfloor d_k/\rho \rfloor$, where each $\mathcal{H}_i^{(k)}$, $1 \leq i \leq p$, is $(3\varepsilon, \rho)$-regular w.r.t. $\mathcal{H}^{(k-1)}$, and where $\mathcal{H}_0^{(k)}$ is $(3\varepsilon, d_k - p\rho)$-regular w.r.t. $\mathcal{H}^{(k-1)}$. In particular, $|\mathcal{H}_0^{(k)}| \leq (d_k - p\rho + 3\varepsilon)|\mathcal{K}_k(\mathcal{H}^{(k-1)})| \leq 2\rho m_1 \cdots m_k$ (since $d_k - \rho \leq p\rho \leq d_k$ and $3\varepsilon \leq \rho$). We establish some notation related to this partition. For $1 \leq i \leq p$, write $\boldsymbol{\mathcal{H}}_i = \{\mathcal{H}^{(1)}, \dots, \mathcal{H}^{(k-1)}, \mathcal{H}_i^{(k)}\}$, write $\mathcal{H}_*^{(k)} = \mathcal{H}_1^{(k)} \cup \dots \cup \mathcal{H}_p^{(k)} = \mathcal{H}^{(k)} \setminus \mathcal{H}_0^{(k)}$, and write $\boldsymbol{\mathcal{H}}_* = \{\mathcal{H}^{(1)}, \dots, \mathcal{H}^{(k-1)}, \mathcal{H}_*^{(k)}\}$ and $\boldsymbol{\mathcal{H}}^{(k-1)} = \{\mathcal{H}^{(1)}, \dots, \mathcal{H}^{(k-1)}\}$.

Now, for $1 \leq i \leq p$, $\boldsymbol{\mathcal{H}}_i$ is a $(3\varepsilon, (d_2, \dots, d_{k-1}, \rho))$-regular (k,k)-complex so that, by Theorem 2.1, all but $\exp\{-c_{\text{Thm. 2.1}}s\}\binom{m_1}{s}$ sets $S \in \binom{V_1}{s}$ render an $(\varepsilon'', (d_2, \dots, d_{k-1}, \rho))$-regular (k,k)-complex $\boldsymbol{\mathcal{H}}_i[\boldsymbol{S}]$. We apply Fact 1.6 to the remainder $\mathcal{H}_0^{(k)}$ to conclude that all but $\exp\{-(\rho^8/6)s\}\binom{m_1}{s}$ sets $S \in \binom{V_1}{s}$ render $|\mathcal{H}_0^{(k)}[\boldsymbol{S}]| \leq 3\rho s m_2 \cdots m_k$. As such, all but

$$\left(\exp\left\{-\tfrac{\rho^8}{6}s\right\} + p\exp\{-c_{\text{Thm. 2.1}}s\}\right)\binom{m_1}{s} \leq \tfrac{2}{\rho}\exp\{-2cs\}\binom{m_1}{s} \leq \exp\{-c\,s\}\binom{m_1}{s}$$

sets $S \in \binom{V_1}{s}$ satisfy all properties immediately above (over all $1 \leq i \leq p$). For the remainder of Case 1, fix such a set $S \in \binom{V_1}{s}$. Statement (2) of Proposition 5.1 guarantees that $\mathcal{H}_*^{(k)}[\boldsymbol{S}]$ is $(p\varepsilon'', p\rho)$-regular w.r.t. $\mathcal{H}^{(k-1)}[\boldsymbol{S}]$, and so $\boldsymbol{\mathcal{H}}_*[\boldsymbol{S}]$ is a $(p\varepsilon'', (d_2, \dots, d_{k-1}, p\rho))$-regular (k,k)-complex. Since $0 < p\varepsilon'' \leq \rho$ and $d_k - \rho \leq p\rho \leq d_k$, we may say, more simply, that $\boldsymbol{\mathcal{H}}_*[\boldsymbol{S}]$ is a $(2\rho, (d_2, \dots, d_k))$-regular (k,k)-complex. We argue that, consequently, $\boldsymbol{\mathcal{H}}[\boldsymbol{S}]$ is an $(\varepsilon', (d_2, \dots, d_k))$-regular (k,k)-complex, and in particular, that $\mathcal{H}^{(k)}[\boldsymbol{S}]$ is (ε', d_k)-regular w.r.t. $\mathcal{H}^{(k-1)}[\boldsymbol{S}]$.

Let $\mathcal{Q}^{(k-1)} \subseteq \mathcal{H}^{(k-1)}[\boldsymbol{S}]$ satisfy $|\mathcal{K}_k(\mathcal{Q}^{(k-1)})| \geq \varepsilon'|\mathcal{K}_k(\mathcal{H}^{(k-1)}[\boldsymbol{S}])|$, where

$$|\mathcal{K}_k(\mathcal{H}^{(k-1)}[\boldsymbol{S}])| \geq (1/2) \prod_{2\leq i\leq k-1} d_i^{\binom{k}{i}} \times s m_2 \cdots m_k \tag{15}$$

follows from Lemma 2.4. (Indeed, since $\boldsymbol{\mathcal{H}}_*[\boldsymbol{S}]$ is a $(2\rho,(d_2,\dots,d_k))$-regular (k,k)-complex, $\boldsymbol{\mathcal{H}}^{(k-1)}[\boldsymbol{S}]$ is a $(2\rho,(d_2,\dots,d_{k-1}))$-regular $(k,k-1)$-complex.) The $(2\rho,(d_2,\dots,d_k))$-regularity of $\boldsymbol{\mathcal{H}}_*[\boldsymbol{S}]$ (recall $\varepsilon' \geq \rho$) implies $|\mathcal{H}_*^{(k)}[\boldsymbol{S}] \cap \mathcal{K}_k(\mathcal{Q}^{(k-1)})| = (d_k \pm 2\rho)|\mathcal{K}_k(\mathcal{Q}^{(k-1)})|$, and so

$$|\mathcal{H}^{(k)}[\boldsymbol{S}] \cap \mathcal{K}_k(\mathcal{Q}^{(k-1)})| = |\mathcal{H}_*^{(k)}[\boldsymbol{S}] \cap \mathcal{K}_k(\mathcal{Q}^{(k-1)})| + |\mathcal{H}_0^{(k)}[\boldsymbol{S}] \cap \mathcal{K}_k(\mathcal{Q}^{(k-1)})|$$

satisfies

$$\begin{aligned}(d_k - 2\rho)|\mathcal{K}_k(\mathcal{Q}^{(k-1)})| &\leq \left|\mathcal{H}^{(k)}[\boldsymbol{S}] \cap \mathcal{K}_k(\mathcal{Q}^{(k-1)})\right| \leq (d_k + 2\rho)|\mathcal{K}_k(\mathcal{Q}^{(k-1)})| + |\mathcal{H}_0^{(k)}[\boldsymbol{S}]| \\ &\leq (d_k + 2\rho)|\mathcal{K}_k(\mathcal{Q}^{(k-1)})| + 3\rho s m_2 \cdots m_k \overset{(15)}{\leq} \Big(d_k + 8\rho \prod_{2\leq i\leq k-1} d_i^{-\binom{k}{i}}/\varepsilon'\Big)|\mathcal{K}_k(\mathcal{Q}^{(k-1)})|\,.\end{aligned}$$

From our choice of ρ, $|\mathcal{H}^{(k)}[\boldsymbol{S}] \cap \mathcal{K}_k(\mathcal{Q}^{(k-1)})| = (d_k \pm \varepsilon')|\mathcal{K}_k(\mathcal{Q}^{(k-1)})|$ follows, concluding Case 1.

Case 2 ($d_k < 1/2$). Consider the (k,k)-complex $\overline{\boldsymbol{\mathcal{H}}} = \{\mathcal{H}^{(1)},\dots,\mathcal{H}^{(k-1)},\mathcal{K}_k(\mathcal{H}^{(k-1)}) \setminus \mathcal{H}^{(k)}\}$, which is $(\varepsilon,(d_2,\dots,d_{k-1},1-d_k))$-regular. By Case 1, all but $\exp\{-c\,s\}\binom{m_1}{s}$ sets $S \in \binom{V_1}{s}$ satisfy that $\overline{\boldsymbol{\mathcal{H}}}[\boldsymbol{S}]$ is an $(\varepsilon',(d_2,\dots,d_{k-1},1-d_k))$-regular (k,k)-complex, or equivalently, that $\boldsymbol{\mathcal{H}}[\boldsymbol{S}]$ is an $(\varepsilon',(d_2,\dots,d_{k-1},d_k))$-regular (k,k)-complex.

□

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