THE ASYMPTOTIC NUMBER OF TRIPLE SYSTEMS NOT CONTAINING A FIXED ONE

BRENDAN NAGLE AND VOJTĚCH RÖDL

Abstract. For a fixed 3-uniform hypergraph $\mathcal{F}$, call a hypergraph $\mathcal{F}$-free if it contains no subhypergraph isomorphic to $\mathcal{F}$. Let $ex(n,\mathcal{F})$ denote the size of a largest $\mathcal{F}$-free hypergraph $\mathcal{G} \subseteq [n]^3$. Let $F_n(\mathcal{F})$ denote the number of distinct labelled $\mathcal{F}$-free $\mathcal{G} \subseteq [n]^3$. We show that $F_n(\mathcal{F}) = 2^{ex(n,\mathcal{F})+o(n^3)}$, and discuss related problems.

1. Introduction

For a finite s-uniform hypergraph $\mathcal{F}$, let $Forb_n(\mathcal{F})$ denote the set of all s-uniform labelled hypergraphs on n vertices which do not contain $\mathcal{F}$ as a subhypergraph. In this paper, we will be interested in estimating the cardinality $F_n(\mathcal{F})$ of the set $Forb_n(\mathcal{F})$.

Let $ex(n,\mathcal{F})$ be the Turán number of $\mathcal{F}$, i.e. the maximum number of edges which an s-uniform hypergraph on n vertices may have while not containing $\mathcal{F}$ as a subgraph. It turns out that the problem of estimating $F_n(\mathcal{F})$ is closely related to the problem of finding Turán numbers $ex(n,\mathcal{F})$.

For $s = 2$, it was proved in [EFR] that

$$F_n(\mathcal{F}) = 2^{ex(n,\mathcal{F})(1+o(1))} = 2^{(1-\frac{1}{\chi(\mathcal{F})-1})\binom{n}{2}(1+o(1))} \tag{1}$$

provided $\chi(\mathcal{F}) > 2$. As noted in [EFR], it seems likely that

$$F_n(\mathcal{F}) = 2^{ex(n,\mathcal{F})(1+o(1))}$$

holds for bipartite $\mathcal{F}$ as well. However, this is not even known for $\mathcal{F} = C_4$, the cycle of length 4. For this case, the best known upper bound $2^{cn^{3/2}}$ is due to Kleitman and Winston [KW].

The aim of this paper is to extend (1) to 3-uniform hypergraphs. Similarly as (1) was proved under the condition that $\mathcal{F}$ is a non-bipartite graph, in the case of 3-uniform hypergraphs we will assume that $\mathcal{F}$ is not a 3-partite hypergraph, i.e. we assume that for any partition $V(\mathcal{F}) = V_1 \cup V_2 \cup V_3$, there is a triple $f \in \mathcal{F}$ and $i \in [3]$ such that $|f \cap V_i| \geq 2$.

Theorem 1.1. *Suppose that $\mathcal{F}$ is a 3-uniform hypergraph which is not 3-partite. Then*

$$F_n(\mathcal{F}) = 2^{ex(n,\mathcal{F})(1+o(1))}.$$

By a well known result of Erdös [E], a 3-uniform hypergraph $\mathcal{F}$ is 3-partite if and only if $\lim_{n\to\infty} \frac{ex(n,\mathcal{F})}{n^3} = 0$. Hence, Theorem 1.1 is a consequence of the following:

Theorem 1.2. *Let $\mathcal{F}$ be an arbitrary 3-uniform hypergraph. Then*

$$F_n(\mathcal{F}) = 2^{ex(n,\mathcal{F})+o(n^3)}.$$

Our approach uses an extension of the method of [EFR] which was based on Szemerédi's Regularity Lemma [S]. Here we use the Hypergraph Regularity Lemma for 3-uniform hypergraphs of [FR] (see also [NR]) and the Counting Lemma of [NR]. Before we give a proof of Theorem 1.2, we need to introduce some necessary concepts preceeding the Hypergraph Regularity Lemma and the Counting Lemma. This will be done in Section 2. The Counting Lemma is explained in Section 3. The proof of Theorem 1.2 is given in Section 4. Finally, in Section 5, we answer a problem raised in [EFR] and discuss possible extensions of Theorem 1.2.

2. The Regularity Lemma and Related Topics

2.1. Graphs and Cylinders.

In this subsection, we provide background definitions and notation used in [NR].

Definition 2.1. We shall refer to any k-partite graph G with k-partition $(V_1, \dots, V_k)$ as a *k-partite cylinder*, and sometimes we will write G as $G = \bigcup_{1\le i<j\le k} G^{ij}$ where $G^{ij} = G[V_i, V_j] = \{\{v_i, v_j\} \in G : v_i \in V_i, v_j \in V_j\}$. If $B \in [k]^3$, then the 3-partite cylinder $G(B) = \bigcup_{\{i,j\}\in[B]^2} G^{ij}$ will be referred to as a *triad*.

We note that a k-partite cylinder is just a graph with a fixed vertex k-partition.

Definition 2.2. Suppose $G \subseteq [V]^2$ is a graph with vertex set $V = V(G)$, and let $X, Y \subseteq V$ be two nonempty disjoint subsets of V. We define the *density* of the pair X, Y with respect to G, denoted $d_G(X, Y)$, as

$$d_G(X,Y) = \frac{|\{x,y\} \in G : x \in X, y \in Y\}|}{|X||Y|}.$$

Definition 2.3. Suppose $G = \bigcup_{1\le i<j\le k} G^{ij}$ is a k-partite cylinder with k-partition $(V_1, \dots, V_k)$ and let $l > 0, \epsilon > 0$ be given. We shall call G an (l, ϵ, k)-*cylinder* provided all pairs V_i, V_j, $1 \le i < j \le k$, induce G^{ij} satisfying that whenever $V_i' \subseteq V_i$, $|V_i'| > \epsilon|V_i|$, and $V_j' \subseteq V_j$, $|V_j'| > \epsilon|V_j|$ then

$$\frac{1}{l}(1-\epsilon) < d_{G^{ij}}(V_i', V_j') < \frac{1}{l}(1+\epsilon).$$

We now define an auxiliary set system pertaining to a k-partite cylinder G.

Definition 2.4. For a k-partite cylinder G, we will denote by $\mathcal{K}_j(G)$, $1 \le j \le k$, that j-uniform hypergraph whose edges are precisely those j-element subsets of $V(G)$ which span cliques of order j in G. Note that the quantity $|\mathcal{K}_j(G)|$ counts the total number of cliques in G of order j, that is, $|\mathcal{K}_j(G)| = |\{X \subseteq V(G) : |X| = j, [X]^2 \subseteq G\}|$.

For an (l, ϵ, k)-cylinder G, the quantity $|\mathcal{K}_k(G)|$ is easy to estimate, as the following fact shows.

Fact 2.5. *For any positive integers k, l, and positive real number θ, there exists ϵ such that whenever G is an (l, ϵ, k)-cylinder with k-partition $(V_1, \dots, V_k), |V_1| = \dots = |V_k| = m$, then*

$$(1-\theta)\frac{m^k}{l^{\binom{k}{2}}} < |\mathcal{K}_k(G)| < (1+\theta)\frac{m^k}{l^{\binom{k}{2}}}. \tag{2}$$

2.2. **3-Uniform Hypergraphs and 3-cylinders.**

In this subsection, we give definitions pertaining to 3-uniform hypergraphs. By a 3-uniform hypergraph $\mathcal{H}$ on vertex set V, we mean $\mathcal{H} \subseteq [V]^3$, thus the members of $\mathcal{H}$ are 3-element subsets of the vertex set V. For all of this paper, we identify the hypergraph $\mathcal{H}$ with the set of its triples.

By a *k-partite* 3-uniform hypergraph $\mathcal{H}$ with k-partition $(V_1, \dots, V_k)$, we understand a hypergraph $\mathcal{H}$ with vertex set V partitioned into k classes $V = V_1 \cup \dots \cup V_k$, where each triple $\{v_x, v_y, v_z\} \in \mathcal{H}$ satisfies that for each $i \in [k]$, $|\{v_x, v_y, v_z\} \cap V_i| \leq 1$.

Definition 2.6. We shall refer to any k-partite, 3-uniform hypergraph $\mathcal{H}$ with k-partition $(V_1, \dots, V_k)$ as a *k-partite 3-cylinder*. For $B \in [k]^3$, we define $\mathcal{H}(B)$ as that subhypergraph of $\mathcal{H}$ induced on $\bigcup_{i \in B} V_i$.

Definition 2.7. Suppose that G is a k-partite cylinder with k-partition $(V_1, \dots, V_k)$, and $\mathcal{H}$ is a k-partite 3-cylinder. We shall say that G *underlies* the 3-cylinder $\mathcal{H}$ if $\mathcal{H} \subseteq \mathcal{K}_3(G)$.

As in Definition 2.4, we define an auxiliary set system pertaining to the 3-cylinder $\mathcal{H}$.

Definition 2.8. If $\mathcal{H}$ is a k-partite 3-cylinder, then for $1 \leq j \leq k$, $\mathcal{K}_j(\mathcal{H})$ will denote that j-uniform hypergraph whose edges are precisely those j-element subsets of $V(\mathcal{H})$ which span a clique of order j in $\mathcal{H}$. Note that the quantity $|\mathcal{K}_j(\mathcal{H})|$ counts the total number of cliques in $\mathcal{H}$ of order j, that is, $|\mathcal{K}_j(\mathcal{H})| = |\{X \subseteq V(\mathcal{H}) : |X| = j, [X]^3 \subseteq \mathcal{H}\}|$.

Definition 2.9. Let $\mathcal{H}$ be a k-partite 3-cylinder with underlying k-partite cylinder $G = \bigcup_{1 \leq i < j \leq k} G^{ij}$, and let $B \in [k]^3$. For the triad $G(B)$, we define the *density* $d_{\mathcal{H}}(G(B))$ of $\mathcal{H}$ with respect to the triad $G(B)$ as

$$d_{\mathcal{H}}(G(B)) = \begin{cases} \frac{|\mathcal{K}_3(G(B)) \cap \mathcal{K}_3(\mathcal{H}(B))|}{|\mathcal{K}_3(G(B))|} & \text{if } |\mathcal{K}_3(G(B))| > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

In other words, the density counts the proportion of triangles of the triad $G(B)$, $B \in [k]^3$, which are triples of $\mathcal{H}$.

More in general, let $Q \subseteq G(B)$, $B \in [k]^3$, where $Q = \bigcup_{\{i,j\} \in [B]^2} Q^{ij}$. One can define the density $d_{\mathcal{H}}(Q)$ of $\mathcal{H}$ with respect to Q as

$$d_{\mathcal{H}}(Q) = \begin{cases} \frac{|\mathcal{K}_3(\mathcal{H}) \cap \mathcal{K}_3(Q)|}{|\mathcal{K}_3(Q)|} & \text{if } |\mathcal{K}_3(Q)| > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

For our purposes, we will need an extension of the definition in (2) above, and will consider a simultaneous density of $\mathcal{H}$ with respect to a fixed r-tuple of triads $\{Q(1), \dots, Q(r)\}$.

Definition 2.10. Let $\mathcal{H}$ be a k-partite 3-cylinder with underlying k-partite cylinder $G = \bigcup_{1\le i<j\le k} G^{ij}$, and let $B \in [k]^3$. Let $\overrightarrow{Q} = \overrightarrow{Q}_B = (Q(1), \dots, Q(r))$ be an r-tuple of triads $Q(s) = \bigcup_{\{i,j\}\in[B]^2} Q^{ij}(s)$ satisfying that for every $s \in [r]$, $\{i,j\} \in [B]^2$, $Q^{ij}(s) \subseteq G^{ij}$. We define the *density* $d_{\mathcal{H}}(\overrightarrow{Q})$ of $\overrightarrow{Q}$ as

$$d_{\mathcal{H}}(\overrightarrow{Q}) = \begin{cases} \frac{|\mathcal{K}_3(\mathcal{H})\cap\bigcup_{s=1}^r \mathcal{K}_3(Q(s))|}{|\bigcup_{s=1}^r \mathcal{K}_3(Q(s))|} & \text{if } |\bigcup_{s=1}^r \mathcal{K}_3(Q(s))| > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

We now give a definition which provides a notion of regularity for 3-cylinders.

Definition 2.11. Let $\mathcal{H}$ be a k-partite 3-cylinder with underlying k-partite cylinder $G = \bigcup_{1\le i<j\le k} G^{ij}$, and let $B \in [k]^3$. Let $r \in \mathbb{N}$ and $\delta > 0$ be given. We say that $\mathcal{H}(B)$ is (δ, r)-*regular with respect to G(B)* if the following regularity condition is satisfied.

Let $\overrightarrow{Q} = \overrightarrow{Q}_B = (Q(1), \dots, Q(r))$ be an r-tuple of triads $Q(s) = \bigcup_{\{i,j\}\in[3]^2} Q^{ij}(s)$, where for all $s \in [r]$, $\{i,j\} \in [B]^2$, $Q^{ij}(s) \subseteq G^{ij}$. Then

$$|\bigcup_{s=1}^{r} \mathcal{K}_3(Q(s))| > \delta|\mathcal{K}_3(G(B))|$$

implies

$$|d_{\mathcal{H}}(\overrightarrow{Q}) - d_{\mathcal{H}}(G(B))| < \delta. \tag{6}$$

If, moreover, it is specified that $\mathcal{H}$ is (δ, r)-regular with respect to G with density $d_{\mathcal{H}}(G) \in (\alpha - \delta, \alpha + \delta)$ for some α, then we say that $\mathcal{H}$ is (α, δ, r)-*regular with respect to* G.

Note that this definition of (δ, r)-regularity was introduced and used in [FR], and then later in [NR].

2.3. **The Regularity Lemma.**

In this section, we state a regularity lemma for 3-uniform hypergraphs established in [FR]. First, we state a number of supporting definitions which can also be found in [FR].

Definition 2.12. Let t be an integer and let V be an n element set. We define an *equitable partition* of V as a partition $V = V_0 \cup V_1 \cup \dots \cup V_t$, where

(*i*) $|V_1| = |V_2| = \dots = |V_t| = \lfloor \frac{n}{t} \rfloor \stackrel{\text{def}}{=} m$,

(*ii*) $|V_0| < t$.

Definition 2.13. Let V be a set. An (l, t, γ, ϵ)-*partition* $\mathcal{P}$ of $[V]^2$ is an (auxilliary) partition $V = V_0 \cup V_1 \cup \dots \cup V_t$ of V, together with a system of edge-disjoint bipartite graphs P^{ij}_α, $1 \le i < j \le t$, $0 \le \alpha \le l_{ij} \le l$, such that

(*i*) $V = V_0 \cup V_1 \cup \dots \cup V_t$ is an equitable partition of V,

(ii) $\bigcup_{\alpha=0}^{l_{ij}} P_\alpha^{ij} = K(V_i, V_j)$ for all i, j, $1 \le i < j \le t$,

(iii) all but $\gamma\binom{t}{2}m^2$ pairs $\{v_i, v_j\}$, $v_i \in V_i$, $v_j \in V_j$, $1 \le i < j \le t$, are edges of ϵ-regular bipartite graphs P_α^{ij}, and

(iv) for all but $\gamma\binom{t}{2}$ pairs i, j, $1 \le i < j \le t$, we have $|P_0^{ij}| \le \gamma m^2$ and

$$\frac{1}{l}(1-\epsilon) \le d_{P_\alpha^{ij}}(V_i, V_j) \le \frac{1}{l}(1+\epsilon)$$

for all $\alpha = 1, \ldots, l_{ij}$.

Definition 2.14. For a 3-uniform hypergraph $\mathcal{H} \subseteq [n]^3$, and $\mathcal{P}$, an (l, t, γ, ϵ)-partition of $[n]^2$, set $m_{\mathcal{P}} = |V_1| = |V_2| = \cdots = |V_t|$. Let P be a triad of $\mathcal{P}$; set further

$$\mu_P = \frac{|\mathcal{K}_3(P)|}{m_{\mathcal{P}}^3}.$$

We say that an (l, t, γ, ϵ)-*partition* $\mathcal{P}$ is (δ, r)-*regular* if

$$\sum \left\{ \mu_P : P \text{ is a } (\delta, r)\text{-irregular triad of } \mathcal{P} \right\} < \delta \left(\frac{n}{m_{\mathcal{P}}} \right)^3.$$

In other words, most of the triples of $\mathcal{H}$ belong to (δ, r)-regular triads of the partition $\mathcal{P}$. We now state the Regularity Lemma of [FR].

Theorem 2.15. *For every δ and γ with $0 < \gamma \le 2\delta^4$, for all integers t_0 and l_0 and for all integer-valued functions $r = r(t, l)$ and all functions $\epsilon(l)$, there exist T_0, L_0, and N_0 such that any 3-uniform hypergraph $\mathcal{H} \subseteq [n]^3$, $n \ge N_0$, admits a $(\delta, r(t, l))$-regular, $(l, t, \gamma, \epsilon(l))$-partition for some t and l satisfying $t_0 \le t < T_0$ and $l_0 \le l < L_0$.*

For future reference, we state the following definition concerning (l, t, γ, ϵ)-regular partitions $\mathcal{P}$ which are (δ, r)-regular with respect to a triple system $\mathcal{H}$.

Definition 2.16. Let $\mathcal{H}$ be a triple system admitting (δ, r)-regular (l, t, γ, ϵ)-partition $\mathcal{P}$. Let $\overrightarrow{s} = \left(s_{\{i,j,k\}_{\alpha\beta\gamma}} \right)$, $1 \le i < j < k \le t$, $1 \le \alpha, \beta, \gamma \le l$, be a vector of integers satisfying

$$d_{\mathcal{H}}(P_\alpha^{ij} \cup P_\beta^{jk} \cup P_\gamma^{ik}) \in [s_{\{i,j,k\}_{\alpha\beta\gamma}}\delta, (s_{\{i,j,k\}_{\alpha\beta\gamma}} + 1)\delta].$$

Then we say that $\overrightarrow{s}$ is the *density vector* of $\mathcal{H}$ over $\mathcal{P}$.

3. The Counting Lemma and the Key Lemma

We begin this section by describing the general setup which we considered in [NR]. Because we will later describe variations of the following environment, we label this first set up as Setup 1.

Setup 1:
For given constants k, δ, l, r and ϵ, and a given set $\{\alpha_B : B \in [k]^3\}$ of positive reals, consider the following setup. Suppose

(*i*) $\mathcal{H}$ is any k-partite 3-cylinder with k-partition $(V_1, \ldots, V_k)$, $|V_1| = \ldots = |V_k| = m$,

(*ii*) let $G = \bigcup_{1 \le i < j \le k} G^{ij}$ be any underlying (l, ϵ, k)-cylinder of $\mathcal{H}$,

(*iii*) suppose further that for all $B \in [k]^3$, $\mathcal{H}(B)$ is (α_B, δ, r)-regular with respect to the triad $G(B)$ (cf. Definition 2.11).

The main result of [NR] was the following Counting Lemma.

Lemma 3.1. The Counting Lemma *Let $k \ge 4$ be a fixed integer. For all sets of positive reals $\{\alpha_B : B \in [k]^3\}$ and constants $\beta > 0$, there exists a positive constant δ so that for all integers $l \ge \frac{1}{\delta}$, there exists r, ϵ such that the following holds: Whenever k-partite 3-cylinder $\mathcal{H}$ and underlying cylinder G satisfy the conditions of Setup 1 with constants k, δ, l, r and ϵ, and set $\{\alpha_B : B \in [k]^3\}$, then*

$$\frac{\Pi_{B\in[k]^3}\alpha_B}{l^{\binom{k}{2}}} m^k(1-\beta) \le |\mathcal{K}_k(\mathcal{H})| \le \frac{\Pi_{B\in[k]^3}\alpha_B}{l^{\binom{k}{2}}} m^k(1+\beta). \tag{7}$$

Note that in what follows, we will only need the following consequence of Lemma 3.1.

Lemma 3.2. *Let $k \ge 4$ be a fixed integer. For all positive reals α and constants $\beta > 0$, there exists a positive constant δ so that for all integers $l \ge \frac{1}{\delta}$, there exists r, ϵ such that the following holds: Whenever k-partite 3-cylinder $\mathcal{H}$ and underlying cylinder G satisfy the conditions of Setup 1 with constants k, δ, l, r and ϵ, and any set $\{\alpha_B : B \in [k]^3\}$ of reals satisfying that for all $B \in [k]^3$, $\alpha_B \ge \alpha$, then*

$$\frac{\Pi_{B\in[k]^3}\alpha_B}{l^{\binom{k}{2}}} m^k(1-\beta) \le |\mathcal{K}_k(\mathcal{H})|\,. \tag{8}$$

For our purposes in this paper, we work with the following weaker setup, which we refer to as Setup 2.

Setup 2:
For given constants k, δ, l, r and ϵ and sets $\{\alpha_B : B \in [k]^3\}$ of nonnegative reals, consider the following setup. Suppose $\mathcal{J} \subseteq [k]^3$ is a triple system, and let $\mathcal{J}^{(2)} \subseteq [k]^2$ be that set of pairs covered by a triple in $\mathcal{J}$. Suppose that triple system $\mathcal{H}$, together with a graph G, satisfy the following conditions:

(*i*) $\mathcal{H}$ is a k-partite 3-cylinder with k-partition $(V_1, \ldots, V_k)$, $|V_1| = \ldots = |V_k| = m$. Moreover, for $B \in [k]^3$, $\mathcal{H}(B) \ne \emptyset$ only if $B \in \mathcal{J}$.

(*ii*) Suppose $G = \bigcup_{\{i,j\}\in\mathcal{J}^{(2)}} G^{ij}$ is an underlying cylinder of $\mathcal{H}$ satisfying that for all $\{i, j\} \in \mathcal{J}^{(2)}$, G^{ij} is an $(l, \epsilon, 2)$-cylinder.

(*iii*) Suppose further that for each $B \in \mathcal{J}$, $\mathcal{H}(B)$ is (α_B, δ, r)-regular with respect to the triad $G(B)$. (cf. Definition 2.11).

We will often refer to the triple system $\mathcal{J}$ as the *cluster triple system of* $\mathcal{H}$.

In their survey paper [KS], Komlós and Simonovits present a lemma for graphs, Theorem 2.1, which they call The Key Lemma. This lemma has been implemented in many applications of the Szemerédi Regularity Lemma for graphs. The following corollary to Lemma 3.2 is an analogous statement for triple systems. For that reason, we preserve the title of Key Lemma.

Corollary 3.3. The Key Lemma *Let $k \geq 4$ be a fixed integer. For all positive reals α, there exists $\delta > 0$ so that for all integers $l \geq \frac{1}{\delta}$, there exists r, ϵ such that the following holds: Whenever k-partite 3-cylinder $\mathcal{H}$, underlying cylinder G, and cluster triple system $\mathcal{J}$ satisfy the conditions of Setup 2 with constants k, δ, l, r and ϵ and any set $\{\alpha_B : B \in [k]^3\}$ of nonnegative reals such that for all $B \in \mathcal{J}$, $\alpha_B \geq \alpha$, then $\mathcal{J} \subseteq \mathcal{H}$ (i.e. $\mathcal{J}$ is a subsystem of $\mathcal{H}$).*

We will show how to derive the Key Lemma from Lemma 3.2. However, first note that essentially the same proof gives the following strengthening of the consequence of Corollary 3.3: For a fixed triple system $\mathcal{J}$, and an integer s, define the *blow up* of $\mathcal{J}$, denoted $\mathcal{J}(s)$, as that triple system obtained by replacing each vertex v of $\mathcal{J}$ with a set $V = V(v)$, of s independent vertices, and replace each triple $\{u, v, w\} \in \mathcal{J}$ with the complete 3-partite triple system $K^{(3)}(U(u), V(v), W(w))$. Then under the same hypothesis of Corollary 3.3, one can conclude not only that $\mathcal{J} \subseteq \mathcal{H}$, but also $\mathcal{J}(s) \subseteq \mathcal{H}$. We now proceed to the definitions of the constants promised by the Key Lemma.

Definitions of the Constants:

Let $k \geq 4$ be an arbitrary integer, and let α be a given positive real. Let $\beta = \frac{1}{2}$. Let

$$\delta = \delta(\alpha, \beta = \frac{1}{2}) \tag{9}$$

be that constant guarenteed to exist by the Lemma 3.2. Let $l > \frac{1}{\delta}$ be any integer. Let

$$r = r(l), \tag{10}$$

$$\epsilon = \epsilon(l) \tag{11}$$

be those constants guarenteed to exist by the Counting Lemma.

We now proceed directly to the proof of the Key Lemma.

Proof of the Key Lemma:

Let $k \geq 4$ be an arbitrary integer and let α be a given positive real. Let $\delta > 0$ be that constant in (9). Let $l > \frac{1}{\delta}$ be an arbitrary integer, and let r and ϵ be given in (10) and (11) respectively. Suppose that for the constants k, α, δ, l, r and ϵ described above and a set $\{\alpha_B : B \in [k]^3\}$ of nonnegative reals, k-partite 3-cylinder $\mathcal{H}$ with cluster triple system $\mathcal{J}$ and underlying system G^{ij}, $\{i, j\} \in \mathcal{J}^{(2)}$, all together satisfy the conditions of Setup 2. We will show that

$$\mathcal{J} \subseteq \mathcal{H}. \tag{12}$$

To that effect, we consider the following procedure on $\mathcal{H}$. For each triple $\{i_1, i_2, i_3\} \in [k]^3$ satisfying $\{i_1, i_2, i_3\} \notin \mathcal{J}$, we replace the empty triple system $\mathcal{H}(\{i_1, i_2, i_3\})$ with the

3-partite 3-cylinder $\mathcal{H}_0(\{i_1,i_2,i_3\}) = \mathcal{K}_3(G(\{i_1,i_2,i_3\})$ (cf. Definition 2.4) on partite sets V_{i_1}, V_{i_2}, V_{i_3}. Note that the triple system $\mathcal{H}_0(\{i_1,i_2,i_3\})$ is trivially $(1,\delta,r)$-regular with respect to the $(l,\epsilon,3)$-cylinder $G^{i_1i_2}\cup G^{i_1i_3}\cup G^{i_2i_3}$. Otherwise, for each triple $\{i_1,i_2,i_3\}\in\mathcal{J}$, let $\mathcal{H}_0(\{i_1,i_2,i_3\}) = \mathcal{H}(\{i_1,i_2,i_3\})$. Let $\mathcal{H}_0$ be the resulting triple system obtained by applying the process above over all $\{i_1,i_2,i_3\}\in[k]^3$.

Similarly, for each pair $\{i,j\}\notin\mathcal{J}^{(2)}$, replace the empty graph G^{ij} with any $(l,\epsilon,2)$-cylinder G_0^{ij} on partite sets V_i, V_j. Otherwise, for any $\{i,j\}\in\mathcal{J}^{(2)}$, set $G_0^{ij}=G^{ij}$. Note that this may be done without effecting any of the triples of $\mathcal{H}$. Let $G_0=\bigcup_{\{i,j\}\in[k]^2}G_0^{ij}$ be the resulting (l,ϵ,k)-cylinder obtained by applying the process above over all $\{i,j\}\in[k]^2$.

After implementing the process described above, we obtain a k-partite 3-cylinder $\mathcal{H}_0$ with underlying (l,ϵ,k)-cylinder G_0. Let the set $\{\overline{\alpha}_B : B\in[k]^3\}$ be defined in the following way: for $B\in\mathcal{J}$, let $\overline{\alpha}_B=\alpha_B$, otherwise, for $B\notin\mathcal{J}$, let $\overline{\alpha}_B=1$. Note that $\mathcal{H}_0$ and G_0 satisfy the conditions in Set Up 1 with the constants k, δ, l, r, ϵ, and the set of positive reals $\{\overline{\alpha}_B : B\in[k]^3\}$ defined above. Though some of the triples of $\mathcal{H}_0$ are not original triples of $\mathcal{H}$, the crucial observation is that the cluster triple system $\mathcal{J}$ still marks original triples of $\mathcal{H}$. If $\{i_1,i_2,i_3\}\in\mathcal{J}$, then $\mathcal{H}_0(\{i_1,i_2,i_3\})=\mathcal{H}(\{i_1,i_2,i_3\})$.

Now we apply Lemma 3.2. By our choice of constants δ, r and ϵ in (9), (10) and (11) respectively, we note that Lemma 3.2 applies (with the constant $\beta=\frac{1}{2}$), and thus conclude

$$|\mathcal{K}_k(\mathcal{H}_0)| \geq \frac{\Pi_{B\in[k]^3}\overline{\alpha}_B}{l^{\binom{k}{2}}}n^k(1-\frac{1}{2}) \geq \frac{\alpha^{|\mathcal{J}|}}{l^{\binom{k}{2}}}\frac{n^k}{2} > 0.$$

We thus conclude that $K_k^{(3)}\subseteq\mathcal{H}_0$. Using the triples $\{i_1,i_2,i_3\}\in\mathcal{J}$, we find a copy $\mathcal{J}\subseteq\mathcal{H}_0$, and due to the observation that $\mathcal{H}_0(\{i_1,i_2,i_3\})=\mathcal{H}(\{i_1,i_2,i_3\})$ on these triples, conclude that $\mathcal{J}\subseteq\mathcal{H}$. $\square$

4. Proof of Theorem 1.2

Recall that in Theorem 1.2, we promised to show that for any triple system $\mathcal{F}$,

$$F_n(\mathcal{F}) = 2^{ex(n,\mathcal{F})+o(n^3)}. \tag{13}$$

To that end, let $\mathcal{F}$ be a fixed triple system. We show the equality in (13). We begin by first showing the easy lower bound for $F_n(\mathcal{F})$ below

$$F_n(\mathcal{F}) \geq 2^{ex(n,\mathcal{F})}. \tag{14}$$

Indeed, the inequality in (14) holds. Let $\mathcal{G}\in Forb_n(\mathcal{F})$ be of size $|\mathcal{G}|=ex(n,\mathcal{F})$. Since each subhypergraph $\mathcal{D}\subseteq\mathcal{G}$ also satisfies $\mathcal{D}\in Forb_n(\mathcal{F})$, it follows that

$$F_n(\mathcal{F}) \geq 2^{ex(n,\mathcal{F})}.$$

Thus, the inequality in (14) is established.

What remains to be shown is the more difficult upper bound. We now show this upper bound. We will show that for any $\nu>0$, the inequality

$$F_n(\mathcal{F}) \leq 2^{ex(n,\mathcal{F})+\nu n^3} \tag{15}$$

holds for sufficiently large n.

To that end, let $\nu > 0$ be given. In order to show (15), we need to first define some auxiliary constants. We define constants σ, ζ, γ, α_0, δ, l_0 and t_0, and functions $r = r(l)$ and $\epsilon = \epsilon(l)$. The following description of these auxiliary constants will be rather technical. Therefore, for the Reader not interested in these details, we give the following hierarchy in (16) below relating these values, and encourage the reader to skip down to the discussion **Proof of (15)** below.

$$\nu \gg \sigma, \zeta \gg \gamma, \alpha_0, \frac{1}{t_0} \gg \delta > \frac{1}{l_0} \gg \frac{1}{r} \gg \epsilon >>> \frac{1}{n}. \tag{16}$$

We note that for convenience, we will denote in our calculations that $o(1)$ is a function of n satisfying $o(1) \to 0$ as $n \to \infty$. Due to the fact that the integer n satisfies that $\frac{1}{n}$ is much smaller than any other value above in (16), we can therefore assume that $o(1)$ is smaller as well.

Definitions of the Auxiliary Constants:

We define auxiliary constants σ, ζ, γ, α_0, δ, l_0 and t_0, and functions $r = r(l)$ and $\epsilon = \epsilon(l)$. For the given constant $\nu > 0$, define

$$\zeta = \zeta(\nu), \tag{17}$$

$$\sigma = \sigma(\nu), \tag{18}$$

$$\theta = \theta(\nu) \tag{19}$$

such that the following inequality

$$(ex(n, \mathcal{F}) + \zeta n^3)(1 + \theta) + o(n^3) + n^3 \sigma \log \frac{1}{\sigma} < ex(n, \mathcal{F}) + \nu n^3 \tag{20}$$

holds for sufficiently large integers n. We note that the function denoted by $o(n^3)$ in (20) above will be seen to be of the order $O(n^2)$, thus the inequality in (20) is easily seen to hold for the appropriate choice of constants ζ, σ and θ, and sufficiently large n.

Next, recall Fact 2.5 which guarentees to every k, l, and θ an $\epsilon > 0$ such that for any (l, ϵ, k)-cylinder G, (2) holds. For $k = 3$, l an arbitrary integer, and θ given in (19) above, let

$$\epsilon_1 = \epsilon_1(l) \tag{21}$$

be the function guarenteed by Fact 2.5.

For the constant ζ given in (17), let

$$t_1 = t_1(\zeta) \tag{22}$$

be an integer satisfying that for all integers n and t, $n \geq t \geq t_1$,

$$\left| \frac{ex(t, \mathcal{F})}{\binom{t}{3}} - \frac{ex(n, \mathcal{F})}{\binom{n}{3}} \right| < \zeta. \tag{23}$$

Note that the existence of t_1 follows from the fact that $\lim_{n\to\infty} \frac{ex(n,\mathcal{F})}{\binom{n}{3}}$ exists for all triple systems $\mathcal{F}$ (cf. [B]).

For the constant σ given in (18), let

$$\gamma = \frac{\sigma}{100}, \tag{24}$$
$$\delta_1 = \frac{\sigma}{100}, \tag{25}$$
$$\alpha_0 = \frac{\sigma}{100}, \tag{26}$$
$$t_2 = \lceil \frac{100}{\sigma} \rceil. \tag{27}$$

Note that as defined, the constants γ, δ_1, α_0 and t_2 satisfy

$$(o(1) + \frac{1}{t_2} + \gamma + 4\delta_1 + \frac{\alpha_0}{3}) < \sigma. \tag{28}$$

Set

$$t_0 = max\{t_1, t_2\} \tag{29}$$

where t_1 and t_2 and in (22) and (27) respectively.

The remainder of our discussion on defining the auxiliary constants will concern the Key Lemma, Corollary 3.3. Recall that regarding the constants involved, Corollary 3.3 states

$$\text{“}\forall k \; \forall \alpha \; \exists \delta : \forall l > \frac{1}{\delta} \; \exists \, r, \epsilon' \text{ so that } \ldots\text{”}.$$

Set

$$k = |V(\mathcal{F})|. \tag{30}$$

For the constant α_0 given in (26), let

$$\delta_2 = \delta_2(k, \alpha_0) \tag{31}$$

be the constant guarenteed by Corollary 3.3. Set

$$\delta = min\{\delta_1, \delta_2\} \tag{32}$$

where δ_1 and δ_2 were given in (25) and (31) respectively.

For any integer $l > \frac{1}{\delta}$, let

$$r = r(l), \tag{33}$$
$$\epsilon' = \epsilon'(l) \tag{34}$$

be those values guarenteed by Corollary 3.3. Set

$$\epsilon = \epsilon(l) = min\{\epsilon_1(l), \epsilon'(l)\} \tag{35}$$

where $\epsilon_1(l)$ and $\epsilon'(l)$ are given in (21) and (34) respectively.

Finally, set

$$l_0 > \frac{1}{\delta} \tag{36}$$

to be any integer, where δ is given in (32). This concludes our definitions of the constants σ, ζ, γ, α_0, δ, l_0 and t_0, and functions $r = r(l)$ and $\epsilon = \epsilon(l)$. We note that with the constants described above, we may assume that the hierarchy in (16) is satisfied.

Proof of (15):

We begin our proof of the upper bound

$$F_n(\mathcal{F}) \leq 2^{ex(n,\mathcal{F})+o(n^3)}$$

as follows. For each $\mathcal{G} \in Forb_n(\mathcal{F})$, we use the Hypergraph Regularity Lemma, Theorem 2.15, to obtain a (δ, r)-regular (l, t, γ, ϵ)-partition $\mathcal{P} = \mathcal{P}_{\mathcal{G}}$. To that end, we must first disclose the input constants and functions required by Theorem 2.15. Recall that regarding the constants involved, Theorem 2.15 states that

$$\text{“}\forall \delta, \gamma, t_0, l_0, r(t,l), \epsilon(l)\ \exists T_0, L_0, N_0 \text{ such that} \ldots\text{”}$$

Let δ be that constant given in (32), γ be that constant given in (24), t_0 be that integer given in (29), and l_0 be that integer given in (36). Further, let $r = r(l)$ be that function given in (33) and $\epsilon = \epsilon(l)$ be that function given in (35).

With the above disclosed input values, Theorem 2.15 guarentees constants T_0, L_0, N_0 so that any triple system $\mathcal{G}$ on $n > N_0$ vertices admits a (δ, r)-regular, (l, t, γ, ϵ)-partition $\mathcal{P}$, for some t and l satisfying $t_0 \leq t \leq T_0$ and $l_0 \leq l \leq L_0$.

We stress here that what is important for us is that provided $n > N_0$, *every* $\mathcal{G} \in Forb_n(\mathcal{F})$ admits a (δ, r)-regular, (l, t, γ, ϵ)-partition $\mathcal{P}_{\mathcal{G}}$. For each $\mathcal{G} \in Forb_n(\mathcal{F})$, choose one (δ, r)-regular, (l, t, γ, ϵ)-partition $\mathcal{P}_{\mathcal{G}}$ guarenteed by Theorem 2.15 and let $\overline{\mathcal{P}} = \{\mathcal{P}_1, \ldots, \mathcal{P}_p\}$ be the set of all such partitions over the family $Forb_n(\mathcal{F})$. Consider an equivalence on $Forb_n(\mathcal{F})$ with classes $C(\mathcal{P}_i, \overrightarrow{s})$ defined by

$$\mathcal{G} \in C(\mathcal{P}_i, \overrightarrow{s}) \iff \begin{cases} \mathcal{P}_i \in \overline{\mathcal{P}} \text{ is the partition associated with } \mathcal{G}, \\ \mathcal{G} \text{ has density vector } \overrightarrow{s} \text{ over } \mathcal{P}_i. \end{cases}$$

Set q to be the number of equivalence classes $C(\mathcal{P}_i, \overrightarrow{s})$. We show that

$$q = 2^{O(n^2)}. \tag{37}$$

Further, for any equivalence class $C(\mathcal{P}_i, \overrightarrow{s})$, we show

$$|C(\mathcal{P}_i, \overrightarrow{s})| = 2^{(ex(n,\mathcal{F})+\zeta n^3)(1+\theta)+o(n^3)+n^3\sigma \log \frac{1}{\sigma}}. \tag{38}$$

Therefore, as a result of (37) and (38), combined with the inequality in (20), we have

$$\begin{aligned} F_n(\mathcal{F}) &= \sum |C(\mathcal{P}_i, \overrightarrow{s})|, \\ &\leq 2^{O(n^2)} 2^{(ex(n,\mathcal{F})+\zeta n^3)(1+\theta)+o(n^3)+n^3\sigma \log \frac{1}{\sigma}}, \\ &= 2^{(ex(n,\mathcal{F})+\zeta n^3)(1+\theta)+o(n^3)+n^3\sigma \log \frac{1}{\sigma}}, \\ &\leq 2^{ex(n,\mathcal{F})+\nu n^3} \end{aligned} \tag{39}$$

where the inequality in (39) follows from the inequality in (20). Thus, we will be finished proving (15) once we have established (37) and (38). We now do this below.

We begin first by estimating the value q.

Estimation of q:

Theorem 2.15 guarentees a partition into at most $\binom{T_0+1}{2}(L_0+1)$ graphs. Since there are at most $\left(\frac{1}{\delta}\right)^{\binom{T_0+1}{3}(L_0+1)^3}$ density vectors which could be associated to any one $\mathcal{P} \in \{\mathcal{P}_1, \dots, \mathcal{P}_p\}$, the number q of classes is at most

$$\left(\frac{1}{\delta}\right)^{\binom{T_0+1}{3}(L_0+1)^3} \left(\binom{T_0+1}{2}(L_0+1)\right)^{\binom{n}{2}} \leq 2^{O(n^2)}.$$

Thus, the bound

$$q = 2^{O(n^2)}$$

is established.

Our second task is to estimate $|C(\mathcal{P}_i, \overrightarrow{s})|$ for any class $C(\mathcal{P}_i, \overrightarrow{s})$. To that effect, fix $C(\mathcal{P}_i, \overrightarrow{s})$, and for simplicity, set $C = C(\mathcal{P}_i, \overrightarrow{s})$ and $\mathcal{P} = \mathcal{P}_i$. For the partition $\mathcal{P}$, let $\mathcal{P}$ have equitable partition $[n] = V_0 \cup V_1 \cup \ldots \cup V_t$, $|V_1| = \ldots = |V_t| = m = \lfloor \frac{n}{t} \rfloor$, and system of bipartite graphs P_α^{ij}, $1 \leq i < j \leq t$, $0 \leq \alpha \leq l_{ij} \leq l$.

Estimation of $|C|$:

Once again, we estimate the number of $\mathcal{G} \in Forb_n(\mathcal{F})$ which are members of the equivalence class C. The main tool we use to count $|C|$ is that the partition $\mathcal{P}$ is a (δ, r)-regular, (l, t, γ, ϵ)-partition with respect to *all* $\mathcal{G} \in C$. Due to the highly regular structure of $\mathcal{P}$, we are able to efficiently count $|C|$, and specifically, show

$$|C| \leq 2^{(ex(n,\mathcal{F})+\zeta n^3)(1+\theta)+o(n^3)+n^3\sigma \log \frac{1}{\sigma}}.$$

To that effect, fix $\mathcal{G} \in C$. We define the following set $\mathcal{E}_0$ to consist of all triples $\{v_i, v_j, v_k\} \in \mathcal{G}$ which fall under any of the following categories:

(i) $\{v_i, v_j, v_k\} \cap V_0 \neq \emptyset$,

(ii) $|\{v_i, v_j, v_k\} \cap V_q| \geq 2$ for some $q \in [t]$,

(iii) $\{v_i, v_j\} \in P_0^{ij}$,

(iv) $\{v_i, v_j\} \in P_\alpha^{ij}$, for some α, $1 \leq \alpha \leq l$, where P_α^{ij} satisfies that either
  (a) P_α^{ij} is ϵ-irregular,
  (b) $d_{P_\alpha^{ij}}(V_i, V_j) \notin \left(\frac{1}{l}(1-\epsilon_2), \frac{1}{l}(1+\epsilon_2)\right)$.

Note that $|\mathcal{E}_0|$ satisfies

$$|\mathcal{E}_0| \leq tn^2 + n\left(\frac{n}{t_0}\right)^2 + \gamma\binom{t}{2}n + 2\gamma\binom{t}{2}\left(\frac{n}{t}\right)^2 n. \tag{40}$$

Further define the set $\mathcal{E}_1$ to consist of all triples $\{v_i, v_j, v_k\} \in \mathcal{G} \setminus \mathcal{E}_0$ which fall under any of the following categories:

(a) $\{v_i, v_j\} \in P^{ij}_\alpha$, $\{v_j, v_k\} \in P^{jk}_\beta$, $\{v_i, v_k\} \in P^{ik}_\gamma$, $1 \le \alpha, \beta, \gamma \le l$, where $P = P^{ij}_\alpha \cup P^{jk}_\beta \cup P^{ik}_\gamma$ is not a (δ, r)-regular triad with respect to $\mathcal{G}(\{i, j, k\})$,

(b) $\{v_i, v_j\} \in P^{ij}_\alpha$, $\{v_j, v_k\} \in P^{jk}_\beta$, $\{v_i, v_k\} \in P^{ik}_\gamma$, $1 \le \alpha, \beta, \gamma \le l$, where $P = P^{ij}_\alpha \cup P^{jk}_\beta \cup P^{ik}_\gamma$ satisfies
$$d_{\mathcal{G}}(P) < \alpha_0.$$

Note that $|\mathcal{E}_1|$ satisfies
$$|\mathcal{E}_1| \le 2\delta t^3 l^3 \left(\frac{n}{t}\right)^3 (1+\theta) + \alpha_0 \binom{t}{3} l^3 \left(\frac{n}{t}\right)^3 \frac{1}{l^3}. \tag{41}$$

Set $\mathcal{E}_{\mathcal{G}} = \mathcal{E}_0 \cup \mathcal{E}_1$. Note that as a result of (40) and (41), $|\mathcal{E}_{\mathcal{G}}|$ satisfies
$$|\mathcal{E}_{\mathcal{G}}| \le (o(1) + \frac{1}{t} + \gamma + 4\delta + \frac{\alpha_0}{3})n^3.$$

Due to (28), we may further bound $|\mathcal{E}_{\mathcal{G}}|$ by
$$|\mathcal{E}_{\mathcal{G}}| \le \sigma n^3.$$

Set $\mathcal{G}' = \mathcal{G} \setminus \mathcal{E}_{\mathcal{G}}$. Thus, upon deleting triples $\mathcal{E}_{\mathcal{G}}$ from $\mathcal{G}$, we delete less than σn^3 triples to obtain the hypergraph $\mathcal{G}'$.

We now construct the cluster hypergraph $\mathcal{J}^C = \mathcal{J}^C(\mathcal{G}) \subseteq [t]^3 \times [l] \times [l] \times [l]$ for $\mathcal{G}'$. We define $\{i, j, k\}_{\alpha\beta\gamma} \in \mathcal{J}^C$, $1 \le i < j < k \le t$, $1 \le \alpha, \beta, \gamma \le l$, if and only if the following conditions are satisfied:

(i) $P = P^{ij}_\alpha \cup P^{jk}_\beta \cup P^{ik}_\gamma$ is a $(l, \epsilon_2, 3)$-cylinder,

(ii) $\mathcal{G}'(\{i, j, k\})$ is $(\overline{\alpha}, \delta, r)$-regular with respect to P, where $\overline{\alpha} \ge \alpha_0$.

We will view the cluster hypergraph $\mathcal{J}^C$ as a multi-set, where for fixed $\{i, j, k\}$, $1 \le i < j < k \le t$, parallel triples
$$\{i, j, k\}_{\alpha\beta\gamma} \in \mathcal{J}^C$$
are possible.

We now state and prove the following fact about the cluster hypergraph $\mathcal{J}^C$.

Fact 4.1. *There exists* $\psi : [t]^2 \longrightarrow [l]$ *such that*
$$\left|\left\{(\{i, j, k\}_{\alpha\beta\gamma} \in \mathcal{J}^C : \psi(\{i, j\}) = \alpha, \psi(\{j, k\}) = \beta, \psi(\{i, k\}) = \gamma, 1 \le i < j < k \le t\right\}\right| \ge \frac{|\mathcal{J}^C|}{l^3}.$$

Proof of Fact 4.1:

We appeal to the Pigeon Hole Principle. Let
$$\Psi = \left\{\psi : [t]^2 \longrightarrow [l]\right\}.$$

Note that
$$|\Psi| = l^{\binom{t}{2}}. \tag{42}$$

Define an auxiliary bipartite graph $G = (V(G), E(G))$ with bipartition
$$V(G) = \Psi \cup \mathcal{J}^C$$
and adjacency rule as follows: we define $\{\psi, \{i,j,k\}_{\alpha\beta\gamma}\} \in E(G)$ if and only if $\psi \in \Psi$, $\{i,j,k\}_{\alpha\beta\gamma} \in \mathcal{J}^C$, and $\psi(\{i,j\}) = \alpha$, $\psi(\{j,k\}) = \beta$, $\psi(\{i,k\}) = \gamma$. Note that for each $\{i,j,k\}_{\alpha\beta\gamma} \in \mathcal{J}^C$
$$deg_G(\{i,j,k\}_{\alpha\beta\gamma}) = l^{\binom{t}{2}-3}.$$
It thus follows that
$$|E(G)| = |\mathcal{J}^C| l^{\binom{t}{2}-3}. \tag{43}$$
On the other hand, due to the equality in (42) and the equality in (43), we infer from the Pigeon Hole Principle that there exists a vertex $\psi_0 \in \Psi$ such that
$$deg_G(\psi_0) \geq \frac{|\mathcal{J}^C| l^{\binom{t}{2}-3}}{l^{\binom{t}{2}}} = \frac{|\mathcal{J}^C|}{l^3}.$$
However, it follows by definition of the graph G that
$$N_G(\psi_0) = \Big\{(\{i,j,k\}_{\alpha\beta\gamma} \in \mathcal{J}^C : \psi(\{i,j\}) = \alpha, \psi(\{j,k\}) = \beta, \psi(\{i,k\}) = \gamma, 1 \leq i < j < k \leq t\Big\}$$
and thus Fact 4.1 follows. □

We thus have the following claim about the cluster hypergraph $\mathcal{J}^C$ which follows from Fact 4.1.

Claim 4.2.
$$|\mathcal{J}^C| \leq l^3 ex(t, \mathcal{F}).$$

Proof of Claim 4.2:

On the contrary, assume
$$|\mathcal{J}^C| > l^3 ex(t, \mathcal{F}). \tag{44}$$
Using Fact 4.1, we see that there exists a function $\psi : [t]^2 \longrightarrow [l]$ such that
$$\Big|\Big\{(\{i,j,k\}_{\alpha\beta\gamma} \in \mathcal{J}^C : \psi(\{i,j\}) = \alpha, \psi(\{j,k\}) = \beta, \psi(\{i,k\}) = \gamma, 1 \leq i < j < k \leq t\Big\}\Big| \geq \frac{|\mathcal{J}^C|}{l^3}.$$
Let $\mathcal{J}$ be the set of triples
$$\mathcal{J} = \Big\{(\{i,j,k\} : \psi(\{i,j\}) = \alpha, \psi(\{j,k\}) = \beta, \psi(\{i,k\}) = \gamma, \text{ and } \{i,j,k\}_{\alpha\beta\gamma} \in \mathcal{J}^C\Big\}$$
and note that $\mathcal{J}$ is a triple system on t vertices without parallel edges.

Under the assumption in (44), it follows that
$$|\mathcal{J}| \geq \frac{|\mathcal{J}^C|}{l^3} > ex(t, \mathcal{F}).$$
Therefore, we immediately see that $\mathcal{F} \subseteq \mathcal{J}$, which leads to a contradiction. Indeed, due to the Key Lemma, Corollary 3.3, $\mathcal{F} \subseteq \mathcal{J}$ implies $\mathcal{F} \subseteq \mathcal{G}' \subseteq \mathcal{G} \in Forb_n(\mathcal{F})$. Thus, to conclude the proof of Claim 4.2, we need only confirm that the Key Lemma applies.

To that end, for the copy of $\mathcal{F}$ in $\mathcal{J}$, let $V_{\mathcal{F}} = V(\mathcal{F})$ be the vertex set of $\mathcal{F}$. Note that as given in (30), $|V(\mathcal{F})| = k$. Let $\mathcal{G}''$ consist of those triples $\{v_{i_1}, v_{i_2}, v_{i_3}\} \in \mathcal{G}'(V_{\mathcal{F}})$ which satisfy that $v_{i_1} \in V_{i_1}$, $v_{i_2} \in V_{i_2}$, $v_{i_3} \in V_{i_3}$ and $\psi(\{v_{i_1}, v_{i_2}\}) = \alpha$, $\psi(\{v_{i_2}, v_{i_3}\}) = \beta$, $\psi(\{v_{i_1}, v_{i_3}\}) = \gamma$, and $\{v_{i_1}, v_{i_2}, v_{i_3}\}_{\alpha\beta\gamma} \in \mathcal{J}^C$. Observe that we may assume that $\mathcal{G}''$ (replacing $\mathcal{H}$) together with cluster triple system $\mathcal{F}$ and system of bipartite graphs $G^{ij} = P^{ij}_{\psi(\{i,j\})}$, $\{i,j\} \in \mathcal{F}^{(2)}$ satisfies the conditions of Setup 2: Indeed, that the properties (i) and (iii) are satisfied for $\mathcal{G}'' \subset \mathcal{G}$ follows from our deletion of the triples $\mathcal{E}_1$. Similarly, (ii) is satisfied by our deletion of the triples of $\mathcal{E}_0$.

By our choice of constants k, α, δ, l, r, ϵ in (30), (26), (32), (33) and (35), it follows that the Key Lemma applies to the triple system $\mathcal{G}''$, the underlying cylinder $\bigcup\{P^{ij}_{\alpha} : \{i,j\} \in [V_{\mathcal{F}}]^2, \alpha = \psi(\{i,j\})\}$, and cluster triple system $\mathcal{F}$. $\Box$

We now bound $|C|$ from above. To that end, define

$$S_C = \bigcup_{\{i,j,k\}_{\alpha\beta\gamma} \in \mathcal{J}_C} \mathcal{K}_3(P^{ij}_{\alpha} \cup P^{jk}_{\beta} \cup P^{ik}_{\gamma}).$$

It follows from Fact 2.5 that for the constant θ in (19) and our choice of ϵ in (35), any $(l, \epsilon, 3)$-cylinder $P^{ij}_{\alpha} \cup P^{jk}_{\beta} \cup P^{ik}_{\gamma}$ on partite sets V_i, V_j, V_k, $|V_i| = |V_j| = |V_k| = m$, satisfies

$$|\mathcal{K}_3(P^{ij}_{\alpha} \cup P^{jk}_{\beta} \cup P^{ik}_{\gamma})| < \frac{m^3}{l^3}(1+\theta). \tag{45}$$

Using Claim 4.2 and (45), we see that $|S_C|$ satisfies

$$\begin{aligned} |S_C| &\leq \frac{m^3}{l^3}(1+\theta)|\mathcal{J}^C|, \\ &\leq m^3 ex(t, \mathcal{F})(1+\theta). \end{aligned} \tag{46}$$

The essential observation now is that for *every* $\mathcal{G} \in C$, $\mathcal{G}' \in S_C$. Thus

$$|\{\mathcal{G}' : \mathcal{G} \in C\}| \leq 2^{n^3 ex(t,\mathcal{F})(1+\theta)}. \tag{47}$$

On the other hand, for each $\mathcal{G} \in C$,

$$\mathcal{G} = \mathcal{G}' \cup \mathcal{E}_{\mathcal{G}},$$

where recall $|\mathcal{E}_{\mathcal{G}}|$ satisfies

$$|\mathcal{E}_{\mathcal{G}}| \leq \sigma n^3.$$

Therefore,

$$|\{\mathcal{E}_{\mathcal{G}}, \mathcal{G} \in C\}| \leq \binom{n^3}{\sigma n^3} \leq 2^{n^3 \sigma \log \frac{1}{\sigma}}. \tag{48}$$

Combining (47) with (48), we infer that

$$|C| \leq 2^{ex(t,\mathcal{F})\frac{n^3}{t^3}(1+\theta) + o(n^3) + n^3\sigma\log(\frac{1}{\sigma})}. \tag{49}$$

The major work in bounding $|C|$ from above is finished. However, to establish (15), we need to replace $ex(t, \mathcal{F})$ by $ex(n, \mathcal{F})$. Recall that we chose t_0 in (29) so that for the auxilliary constant $\zeta > 0$, for all $n \geq t \geq t_0$,

$$\left| \frac{ex(t,\mathcal{F})}{\binom{t}{3}} - \frac{ex(n,\mathcal{F})}{\binom{n}{3}} \right| < \zeta$$

or equivalently

$$\left| \frac{ex(t,\mathcal{F})}{\binom{t}{3}} \binom{n}{3} - ex(n,\mathcal{F}) \right| < \zeta \binom{n}{3}.$$

The above inequality implies

$$ex(t,\mathcal{F}) \frac{n^3}{t^3} \leq ex(n,\mathcal{F}) + \zeta n^3. \tag{50}$$

Combining (49) with (50) implies

$$|C| = 2^{(ex(n,\mathcal{F})+\zeta n^3)(1+\theta(\epsilon_2))+o(n^3)+n^3\sigma\log(\frac{1}{\sigma})}.$$

Therefore, our proof of (15) is complete.

5. Remarks

By Theorem 1.1, the problem of estimating $F_n(\mathcal{F})$ for non 3-partite triple systems $\mathcal{F}$ is essentially equivalent to finding the Turán number $ex(n,\mathcal{F})$. For $s > 2$, the problem of determining Turán numbers, however, is notoriously open, and $\lim_{n\to\infty} \frac{ex(n,\mathcal{F})}{n^3}$ is unknown for nearly all non 3-partite 3-uniform hypergraphs $\mathcal{F}$.

For $s = 2$, the situation is much clearer due to well known results of Turán [T] and Erdös and Stone [ES].

The complete r-partite graph $K(X_1, \ldots, X_r)$ consists of all edges connecting distinct X_i and X_j. Note that this graph contains no K_{r+1} and has chromatic number r if $X_i \neq \emptyset$ for all $i \in [r]$. To maximize $|K(X_1, \ldots X_r)|$, one chooses the X_i to have as equal sizes as possible, i.e. $\lfloor \frac{n}{r} \rfloor \leq |X_i| \leq \lceil \frac{n}{r} \rceil$. Then Turán's Theorem states

Theorem 5.1.

$$ex(n, K_{r+1}) = |K(X_1, \ldots, X_r)| = \binom{n}{2}(1 - \frac{1}{r} + o(1)).$$

Let $\chi(\mathcal{F})$ denote the chromatic number of $\mathcal{F}$. An old result of Erdös, Stone, and Simonovits shows that $ex(n,\mathcal{F})$ and $ex(n, K_{\chi(\mathcal{F})})$ are closely related:

Theorem 5.2. *Set $\chi(\mathcal{F}) = r$, $r \geq 3$. Then*

$$ex(n, K_r) \leq ex(n,\mathcal{F}) \leq (1 + o(1))ex(n, K_r).$$

The following extension of Theorem 5.2 was proved in [EFR].

Theorem 5.3. *Let ϵ_0 be an arbitrary positive number and $\mathcal{G}$ an $\mathcal{F}$-free graph on n vertices. Then for $n \geq n_0(\epsilon_0, \mathcal{F})$, one can remove less than $\epsilon_0 n^2$ edges from $\mathcal{G}$ so that the remaining graph is K_r-free, where $r = \chi(\mathcal{F})$.*

Unfortunately, in the case of 3-uniform hypergraphs, there is no extension of Theorem 5.1 and Theorem 5.2 known. One can however give an extension to Theorem 5.3.

Definition 5.4. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are two triple systems. We say a function $\psi : V(\mathcal{F}_1) \longrightarrow V(\mathcal{F}_2)$ is a *homomorphism* if for every $\{a,b,c\} \in \mathcal{F}_1$, $\{\psi(a),\psi(b),\psi(c)\} \in \mathcal{F}_2$. We say that $\mathcal{F}_2$ is a *proper image* of $\mathcal{F}_1$ if ψ is not 1-1 and if every $f \in \mathcal{F}_2$ is an image of some triple of $\mathcal{F}_1$.

Theorem 5.5. *Let $\mathcal{F}$ be a fixed triple system, and let $\mathcal{F}'$ be any homomorphic image of $\mathcal{F}$. For any $\epsilon > 0$, there exists $n_0 = n_0(\epsilon, \mathcal{F}, \mathcal{F}')$ such that if $\mathcal{G} \in Forb_n(\mathcal{F})$, $n \geq n_0$, then there exists a set $\mathcal{E} \subseteq \mathcal{G}$, $|\mathcal{E}| < \epsilon n^3$, such that $\mathcal{G} \setminus \mathcal{E} \in Forb_n(\mathcal{F}')$.*

We remark that Theorem 5.5 may be proved along the same lines as Theorem 1.2.

Let $K_t(l,s)$ be the complete t-partite l-uniform hypergraph on partite sets $V_1 \cup \ldots \cup V_t$, $|V_1| = \ldots = |V_t| = s$. That is,

$$K_t(l,s) = \Big\{\{v_{i_1}, \ldots, v_{i_l}\} : v_{i_j} \in V_{i_j}, 1 \leq i_j \leq t, 1 \leq j \leq l\Big\}$$

For $s = 1$, we write $K_t(l,1)$ simply as $K_t^{(l)}$. The following problem was raised in Problem 6.1 of [EFR].

Problem 5.6. *Let integers t, l, $t \geq l \geq 3$, be given, along with arbitrary integer s and positive real ϵ. Suppose $\mathcal{G} \subseteq [n]^l$ is a $K_t(l,s)$-free hypergraph on $n > n_0(t,l,s,\epsilon)$ vertices. Is it possible to remove less that ϵn^l edges of $\mathcal{G}$ to obtain a $K_t^{(l)}$-free hypergraph?*

Note that Theorem 5.5 provides an answer to Problem 5.6 for $l = 3$. Indeed, set $\mathcal{F} = K_t(3,s)$ and $\mathcal{F}' = K_t^{(3)}$. Since $K_t^{(3)}$ is a homomorphic image of $K_t(3,s)$, the statement of Theorem 5.5 immediately answers Problem 5.6 affirmatively.

From Theorem 5.5, we also have the following corollary.

Corollary 5.7. *Let $\mathcal{F}'$ be a homomorphic image of $\mathcal{F}$. Then*

$$ex(n,\mathcal{F}') \geq ex(n,\mathcal{F}) - o(n^3).$$

It is conceivable that equality can hold above in the following sense: call an s-uniform hypergraph $\mathcal{G}$ *irreducible* if every pair of vertices is covered by an edge of $\mathcal{G}$. We formulate the following question.

Question 5.8. *Is it true that any 3-uniform hypergraph $\mathcal{F}$ which is not irreducible has a proper image $\mathcal{F}'$ so that*

$$ex(n,\mathcal{F}) = ex(n,\mathcal{F}') + o(n^3)$$

holds?

The above question can be reformulated and generalized as follows:

Question 5.9. *Is it true that any s-uniform hypergraph $\mathcal{F}$ has an irreducible image $\mathcal{F}'$ so that*

$$ex(n,\mathcal{F}') = ex(n,\mathcal{F}) + o(n^s)$$

holds?

If true, this would give an extension of Theorem 5.2 to s-uniform hypergraphs.

As a final remark, one may extend Theorem 1.2 forbidding more than just one hypergraph $\mathcal{F}$. Let $\{\mathcal{F}_i\}_{i\in I}$ be a set of s-uniform hypergraphs and let $Forb_n(\{\mathcal{F}_i\}_{i\in I})$ be the set of all s-uniform hypergraphs on n vertices not containing any $\mathcal{F} \in \{\mathcal{F}_i\}_{i\in I}$ as a subhypergraph. Set $F_n(\{\mathcal{F}_i\}_{i\in I}) = |Forb_n(\{\mathcal{F}_i\}_{i\in I})|$ and $ex(n, \{\mathcal{F}_i\}_{i\in I})$ to be the Turán number for the class $\{\mathcal{F}_i\}_{i\in I}$, that is,

$$ex(n, \{\mathcal{F}_i\}_{i\in I}) = \max\left\{|\mathcal{G}| : \mathcal{G} \subseteq [n]^s \text{ is } \mathcal{F}\text{-free for all } \mathcal{F} \in \{\mathcal{F}_i\}_{i\in I}\right\}.$$

By using the proof of Theorem 1.2, one may analogously establish the following theorem.

Theorem 5.10. *For any set of triple systems $\{\mathcal{F}_i\}_{i\in I}$,*

$$|F_n(\{\mathcal{F}_i\}_{i\in I})| = 2^{ex(n,\{\mathcal{F}_i\}_{i\in I})+o(n^3)}.$$

The following related problem was considered by Alekseev in [A] and Bollobás and Thomason in [BT]. Let $\{\mathcal{F}_i\}_{i\in I}$ be a set of s-uniform hypergraphs and let $ForbInd_n(\{\mathcal{F}_i\}_{i\in I})$ be the set of all s-uniform hypergraphs on n vertices not containing any $\mathcal{F} \in \{\mathcal{F}_i\}_{i\in I}$ as an *induced* subhypergraph, and set $FI_n(\{\mathcal{F}_i\}_{i\in I}) = |ForbInd_n(\{\mathcal{F}_i\}_{i\in I})|$. Observe that the class $ForbInd(\{\mathcal{F}_i\}_{i\in I}) = \bigcup_{n=1}^{\infty} ForbInd_n(\{\mathcal{F}_i\}_{i\in I})$ is closed under taking induced subhypergraphs (Bollobás and Thomason [BT] call such classes *hereditary*). For an integer n and a given class $\{\mathcal{F}_i\}_{i\in I}$ of s-uniform hypergraphs, consider sets $\mathcal{M}, \mathcal{N} \subseteq [n]^s$ with the following two properties:

- (i) $\mathcal{M} \cap \mathcal{N} = \emptyset$.
- (ii) For $\mathcal{G} = [n]^s \setminus (\mathcal{M} \cup \mathcal{N})$, for all $\mathcal{G}' \subseteq \mathcal{G}$, $\forall \mathcal{F} \in \{\mathcal{F}_i\}_{i\in I}$, $\mathcal{F}$ is not an induced subhypergraph of $\mathcal{G}' \cup \mathcal{M}$.

Define

$$ex_{ind}(n, \{\mathcal{F}_i\}_{i\in I}) = \max\{|[n]^s \setminus (\mathcal{M} \cup \mathcal{N})| : \mathcal{M},\ \mathcal{N} \text{ have the properties in } (i) \text{ and } (ii) \text{ above }\}.$$

For $s = 2$, Bollobás and Thomason [BT] used Szemerèdi's Regularity Lemma to show that for all sets of graphs $\{\mathcal{F}_i\}_{i\in I}$, if one writes

$$FI_n(\{\mathcal{F}_i\}_{i\in I}) = 2^{c_n\binom{n}{2}},$$

then

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{ex_{ind}(n, \{\mathcal{F}_i\}_{i\in I})}{\binom{n}{2}} = 1 - \frac{1}{r-1}, \tag{51}$$

where $r = r(\{\mathcal{F}_i\}_{i\in I})$ is an integer valued function of $\{\mathcal{F}_i\}_{i\in I}$.

Together with Y. Kohayakawa in the upcoming paper [KNR], the present authors extend (51) for $s = 3$ to say that for all sets of triple systems $\{\mathcal{F}_i\}_{i\in I}$,

$$FI_n(\{\mathcal{F}_i\}_{i\in I} = 2^{ex_{ind}(n,\{\mathcal{F}_i\}_{i\in I})+o(n^3)}.$$

References

[A] V. E. Alekseev, *On the entropy values of hereditary classes of graphs*, Discrete Math. Appl. 3 (1993) 191-199.

[B] B. Bollobás, *Combinatorics: set systems, hypergraphs, families of vectors, and combinatorial probability*, Cambridge University Press, (1986).

[BT] B. Bollobás, A. Thomason, *Hereditary and Monotone Properties of Graphs*, The Mathematics of Paul Erdös. R. L. Graham, J. Nešetřil (eds), Algorithms and Combinatorics, Vol 13, Springer Verlag (1997).

[E] P. Erdös, *On extremal problems of graphs and generalized graphs*, Israel J. of Math. 2, 189-190 (1965)

[EFR] P. Erdös, P. Frankl, V. Rödl, *The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs have no exponent*, Graphs and Combinatorics 2, 113-121 (1986)

[ES] P. Erdös, A. H. Stone, *On the structure of linear graphs*,J Bull. Amer. Math. Soc., 52, 1087-1091 (1946)

[FR] P. Frankl, V. Rödl, *Extremal problems on set systems*, submitted.

[KNR] Y. Kohayakawa, B. Nagle, V. Rödl, *Hereditary properties of triple systems*, manuscript

[KS] J. Komlós, M. Simonovits, *Szemerédi's Regularity Lemma and its application in graph theory*, In: Combinatorics, Paul Erdös is eighty, Volume 2, János Bolyai Mathematical Society, Budapest, Hungary, 295-352 (1996)

[KW] D. J. Kleitman, K. J. Winston, *The asymptotic number of lattices*, In: Combinatorial Mathematics, Optimal Designs and their Applications, edited by J. Shrivastava, Ann. Discrete Math. 6, 243-249, (1980)

[NR] B. Nagle, V. Rödl, *Regularity properties for triple systems*, manuscript

[S] E. Szemerédi, *Regular partitions of graphs*, In: Proc. Colloq. Int. CNRS, 399-401, Paris CNRS 1976

[T] P. Turán, *An extremal problem in graph theory (in Hungarian)*, Mat. Fiz. Lapok 48, 436-452 (1941)

Georgia Institute of Technology, Atlanta, GA, 30032, USA
E-mail address: nagle@math.gatech.edu

Emory University, Atlanta, GA, 30032, USA
E-mail address: rodl@mathcs.emory.edu