# An Algorithmic Version of the Hypergraph Regularity Method

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## Abstract

*Extending the Szemerédi Regularity Lemma for graphs, P. Frankl and the third author [11] established a 3-graph Regularity Lemma guaranteeing that all large triple systems admit partitions of their edge sets into constantly many classes where most classes consist of regularly distributed edges. Many applications of this lemma require a companion Counting Lemma [26] allowing one to estimate the number of copies of $K_k^{(3)}$ in a "dense and regular" environment created by the 3-graph Regularity Lemma. Combined applications of these lemmas are known as the 3-graph Regularity Method. In this paper, we provide an algorithmic version of the 3-graph Regularity Lemma which, as we show, is compatible with a Counting Lemma. We also discuss some applications.*

*For general k-uniform hypergraphs, Regularity and Counting Lemmas were recently established by Gowers [16] and by Nagle, Rödl, Schacht, and Skokan [27, 35]. We believe the arguments here provide a basis toward a general algorithmic hypergraph regularity method.*

## 1. Introduction

Szemerédi's Regularity Lemma [38] for graphs is one of the most powerful tools in combinatorics, with applications ranging across combinatorial number theory, extremal graph theory and theoretical computer science (see [24] for an excellent survey of applications).

The great importance of Szemerédi's Regularity Lemma has led to a search for extensions to $k$-uniform hypergraphs, for example [3, 6, 10, 12, 13]. While these early generalizations did lead to some interesting applications, they did not seem to capture the full power of Szemerédi's lemma for graphs. In particular, they did not allow for the embedding of small subsystems within a regular structure. The first generalized regularity lemma that did have this property was the lemma of Frankl and Rödl [11] for 3-uniform hypergraphs (3-graphs). In what follows we refer to this result as the *3G-Lemma*, for short. It guarantees that any large 3-graph $\mathcal{G}$ admits a bounded partition of its triples, most classes of which are "regularly distributed". The embedding of small subsystems is made possible by a companion result, called the Counting Lemma [26], which together with the 3G-Lemma has resulted in various applications to hypergraph problems [5, 18, 22, 21, 26, 25, 30, 31, 36].

The original proof of Szemerédi's Regularity Lemma for graphs was not algorithmic. An algorithmic version of Szemerédi's Lemma was later established in [1, 2, 9] by Alon, Duke, Lefmann, Rödl and Yuster, rendering constructive solutions to many problems where Szemerédi's Lemma is applied (see [2] for applications). Our object is to establish compatible algorithmic versions of the hypergraph regularity lemma and the Counting Lemma. Algorithmic versions of earlier hypergraph regularity lemmas have been established, including important work of Frieze and Kannan [12, 13] (cf. [6]), but since these do not allow for the embedding of small substructures, their power is restricted to applications not requiring this feature.

Extending the 3G-Lemma, regularity lemmas and counting lemmas for $k$-uniform hypergraphs, also allowing the embedding of small substructures, were developed later by Gowers [16, 17] and Nagle, Rödl, Schacht and Skokan [27, 35] (for a recent survey on these topics see [29]). We refer to the use of a regularity lemma and a counting lemma together as the Hypergraph Regularity Method. An immediate application of the hypergraph regularity method proves a so-called 'Removal Lemma' (cf. [16, 27, 34, 39]). This lemma, in turn, allows alternative proofs (cf. [11, 33]) of partition theorems originally obtained by Szemerédi, Furstenberg and Katznelson [14, 15, 37], including Sze-

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merédi's celebrated Density Theorem: *for fixed $\delta > 0$ and integer $k$, every subset $Z \subseteq \{1, \ldots, n\}$, $n > n_0(\delta, k)$, of size $|Z| > \delta n$, contains an arithmetic progression of length $k$.* Very recently, a hypergraph regularity lemma was developed by Tao [39] for some striking number theory applications. Other applications of the regularity method to $k$-uniform hypergraphs can be found in [28, 29, 32, 33, 34].

Since the 3G-Lemma has already led to many applications, and also to minimize technical details, we will restrict our attention here to the 3-graph case. Our work here completes a project begun in Dementieva, Haxell, Nagle and Rödl [7], and we will require a main result of [7] for a crucial step in our proof (see Section 4). It appears that our argument here uses some similar ideas to those in the regularity lemma for 3-uniform hypergraphs independently obtained by Gowers [17], and in this way it is likely closer to the work of Gowers than to the work on $k$-uniform hypergraph regularity by Nagle, Rödl, Schacht and Skokan [27, 35]. We believe that there should be no principle obstacle to developing a completely algorithmic regularity method for general $k$. Gowers' [16] version of his lemma for $k$-uniform hypergraphs reinforces our belief.

The original 3G-Lemma [11] is based on a notion of hypergraph regularity called *$(\delta, r)$-regularity* which we define in Definition 7 of this paper. As can be seen in Definition 7, $(\delta, r)$-regularity has a somewhat technical definition, and verifying it is certainly a co-NP-complete problem. While this fact in itself is not necessarily an insurmountable problem in developing an algorithmic version (see e.g. [2]), here our algorithmic lemma does not attempt to capture fully the property of $(\delta, r)$-regularity. It uses instead a weaker notion, which is nevertheless sufficient to provide algorithmic proofs for a number of existing applications of the 3-graph regularity method, and to provide new ones.

In the next section we describe some algorithmic applications of our main results. In Section 3, we state our main results and we sketch their proofs in Section 4. Some details are omitted in this extended abstract due to space limitations. The full details will appear in a complete paper version [19].

## 2. Applications

The algorithmic version of Szemerédi's Regularity Lemma lead to many applications to constructive graph problems (cf. [1, 2]). The work of our paper enables most of these graph applications to be extended to analogous statements for 3-graphs. In this section, we discuss two applications of our work, and refer the reader to [19] for other applications.

Our first example considers algorithmic enumeration of a fixed 3-graph $\mathcal{J}_0$ appearing as a subsystem of a given 3-graph $\mathcal{G}$. To that end, let $\mathcal{J}_0(\mathcal{G})$ denote the family of copies of $\mathcal{J}_0$ appearing as subsystems of $\mathcal{G}$ and let $\mathcal{J}_0^{\mathrm{ind}}(\mathcal{G})$ denote the family of copies of $\mathcal{J}_0$ appearing as *induced* subsystems of $\mathcal{G}$. We are interested in estimating $|\mathcal{J}_0(\mathcal{G})|$ and $|\mathcal{J}_0^{\mathrm{ind}}(\mathcal{G})|$ for arbitrary but fixed $\mathcal{J}_0$ and arbitrary but large $\mathcal{G}$. Clearly, these parameters could be computed precisely in time $N^k$, where $N$ denotes the number of vertices of $\mathcal{G}$. However, with the methods developed here, we can approximate these parameters, up to an error of $o(N^k)$, in time $O(N^6)$.

**Theorem 1** *Let integers $k \geq 3$ and $\eta > 0$ and 3-graph $\mathcal{J}_0$ on $k$ vertices be given. Then there exists $N_0 = N_0(k, \eta)$ such that for any 3-graph $\mathcal{G}$ with $N > N_0(k, \eta)$ vertices, the quantities $|\mathcal{J}_0(\mathcal{G})|$ and $|\mathcal{J}_0^{\mathrm{ind}}(\mathcal{G})|$ may be approximated, in time $O(N^6)$, within an additive error of $\eta N^k$.*

Theorem 1 is actually a straightforward consequence of the algorithmic methods we establish in this paper. For illustrative purposes, however, we sketch the proof of how Theorem 1 follows from our methods in Section 5.

Our next example considers a hypergraph packing approximation scheme. Let a fixed 3-graph $\mathcal{F}$ be given. For a 3-graph $\mathcal{G}$, an *$\mathcal{F}$-packing* of $\mathcal{G}$ is a collection of pairwise triple-disjoint copies $\mathcal{F}_0$ of $\mathcal{F}$ contained in $\mathcal{G}$. We denote by $\nu_{\mathcal{F}}(\mathcal{G})$ the maximum size of an $\mathcal{F}$-packing of $\mathcal{G}$. Setting $\binom{\mathcal{G}}{\mathcal{F}}$ to be the set of copies $\mathcal{F}_0$ of $\mathcal{F}$ contained in $\mathcal{G}$, a *fractional $\mathcal{F}$-packing* of $\mathcal{G}$ is any function $\psi : \binom{\mathcal{G}}{\mathcal{F}} \to [0, 1]$ such that for every fixed edge $e \in \mathcal{G}$, $\sum_{\binom{\mathcal{G}}{\mathcal{F}} \ni \mathcal{F}_0 \ni e} \psi(\mathcal{F}_0) \leq 1$. Then, $\nu^*_{\mathcal{F}}(\mathcal{G})$ is defined to be the maximum value of $\sum_{\mathcal{F}_0 \in \binom{\mathcal{G}}{\mathcal{F}}} \psi(\mathcal{F}_0)$ taken over all fractional $\mathcal{F}$-packings of $\mathcal{G}$. It is not hard to see that $\nu^*_{\mathcal{F}}(\mathcal{G}) \geq \nu_{\mathcal{F}}(\mathcal{G})$ holds for all 3-graphs $\mathcal{G}$.

While computing $\nu_{\mathcal{F}}(\mathcal{G})$ is an NP-hard problem (cf. [8]), computing $\nu^*_{\mathcal{F}}(\mathcal{G})$ is a linear programming problem (and hence can done in polynomial time). Extending a result of the first and third author [20] for graphs, the current authors proved in [18] that

$$\nu^*_{\mathcal{F}}(\mathcal{G}) - \nu_{\mathcal{F}}(\mathcal{G}) = o(|V(\mathcal{G})|^3) \tag{1}$$

holds for all 3-graphs $\mathcal{G}$. Quite recently, the asymptotic (1) was extended to $k$-graphs by Schacht, Siggers, Tokushige and the third author in [32] using the hypergraph regularity method.

The results of this paper may be combined with the arguments of [18] to give the following constructive extension of (1). We give the details in a forthcoming paper.

**Theorem 2** *For all fixed 3-graphs $\mathcal{F}$ and constants $\varepsilon > 0$, there exists $N_0$ so that for all 3-graphs $\mathcal{G}$ on $N > N_0$ vertices, an $\mathcal{F}$-packing of $\mathcal{G}$ of size $\nu_{\mathcal{F}}(\mathcal{G}) - \varepsilon N^3$ can be constructed in polynomial time.*

## 3. Algorithmic 3G-Method

We begin our discussion by reviewing the notion of an $\varepsilon$-regular pair (as in Szemerédi's Lemma). A bipartite graph $H = (X \cup Y, E)$ is $(d, \varepsilon)$-*regular* if for every $X' \subseteq X$, $|X'| > \varepsilon|X|$, and $Y' \subseteq Y$, $|Y'| > \varepsilon|Y|$, we have $|d_H(X', Y') - d| < \varepsilon$ where $d_H(X', Y') = |H[X', Y']||X'|^{-1}|Y'|^{-1}$ is the *density* of the bipartite graph $H[X', Y']$ induced on the sets $X'$ and $Y'$. The pair $X, Y$ is $\varepsilon$-*regular* if it is $(d, \varepsilon)$-regular for some $d$.

The notion of $\varepsilon$-regularity in bipartite graphs is a *global* property, in the sense that it asserts a fact about all reasonably large subsets of $X$ and $Y$. In the algorithmic version of Szemerédi's Lemma [2], the key idea is that $\varepsilon$-regularity is essentially equivalent to an easily verifiable *local* property, depending only on the sizes of the vertex neighborhoods and the pairwise intersections of vertex neighborhoods. This local property implies another equivalent property (see e.g. [23]), that a bipartite graph is $\varepsilon$-regular if it has close to the minimum possible number of 4-cycles amongst all bipartite graphs of its size and density. (Similar ideas capturing global properties with local ones were used earlier in e.g. [4, 40]). Here we focus on a similar local property of 3-graphs $\mathcal{H}$ based on the number of copies of the fixed 3-graph $K^{(3)}_{2,2,2}$ in $\mathcal{H}$, where $K^{(3)}_{2,2,2}$ denotes the complete 3-partite 3-graph with 2 vertices in each class (see Definition 3 below).

Szemerédi's Lemma [38] states that, given $\varepsilon > 0$ and $t_0$, there exists $T_0 = T_0(\varepsilon, t_0)$, such that every large enough graph $G$ has a vertex partition into $t$ almost equal parts $V_i$, such that $t_0 \leq t \leq T_0$, and all but at most $\varepsilon t^2$ of the pairs $(V_i, V_j)$ induce $\varepsilon$-regular subgraphs of $G$. In other words, $G$ has a partition of its underlying vertex set into a constant number of parts, such that almost all the edges of $G$ lie in subgraphs that are "regularly distributed" between two parts. Analogously, given a 3-graph $\mathcal{G}$ with vertex set $V$, the 3G-Lemma guarantees a partition $\mathcal{P}$ of the underlying set, which this time consists of both the vertex set $V$ and the set of pairs $\binom{V}{2}$, with respect to which $\mathcal{G}$ behaves 'regularly'.

The type of partitions we consider, called $(\ell, t, \varepsilon)$-*partitions*, have the following form.

1. $\mathcal{P}$ has auxiliary vertex partition $V = V_1 \cup \cdots \cup V_t$ satisfying $|V_1| \leq \cdots \leq |V_t| \leq |V_1| + 1$;

2. $\mathcal{P}$ partitions each $K(V_i, V_j) = \bigcup_{a=1}^{\ell} P^{ij}_a$, $1 \leq i < j \leq t$, into bipartite graphs $P^{ij}_a$, each of which is $(1/\ell, \varepsilon)$-regular. Here $K(V_i, V_j)$ denotes the complete bipartite graph with vertex bipartition $V_i \cup V_j$.

For an $(\ell, t, \varepsilon)$-partition $\mathcal{P}$, any 3-partite graph $P$ of the form $P = P^{ij}_a \cup P^{jk}_b \cup P^{ik}_c$, where $1 \leq i < j < k \leq t$, $1 \leq a, b, c \leq \ell$, is called a *triad* of $\mathcal{P}$. Denote by $\mathrm{Triad}(\mathcal{P})$ the set of all such triads $P$.

In the 3G-Lemma, the triads of a partition $\mathcal{P}$ play the same role as the pairs of vertex classes in Szemerédi's Lemma. Thus in particular we will have a notion of *density* of a 3-graph with respect to an underlying triad of $\mathcal{P}$. For a vertex-pair partition $\mathcal{P}$ of $V$ and for $P \in \mathrm{Triad}(\mathcal{P})$, let $\mathcal{K}^{(2)}_3(P)$ denote the system of *triangles* of $P$:

$$\mathcal{K}^{(2)}_3(P) = \Big\{ \{x, y, z\} \in \binom{V}{3} : \{x, y, z\} \text{ induces a triangle } K^{(2)}_3 \text{ in } P \Big\}.$$

Now, let $\mathcal{G}$ be a 3-graph on the vertex set $V$. For $P \in \mathrm{Triad}(\mathcal{P})$, we write $\mathcal{G}_P = \mathcal{G} \cap \mathcal{K}^{(2)}_3(P)$ and define the *density* of $\mathcal{G}_P$ with respect to $P$ as $\alpha_P = d_{\mathcal{G}_P}(P) = |\mathcal{G}_P|/|\mathcal{K}^{(2)}_3(P)|$. Set

$$\mathcal{K}^{(3)}_{2,2,2}(\mathcal{G}_P) = \Big\{ J \in \binom{V}{6} : J \text{ induces a copy of } K^{(3)}_{2,2,2} \text{ in } \mathcal{G}_P. \Big\}$$

We now arrive at our central definition, Definition 3 below. Let triad $P \in \mathrm{Triad}(\mathcal{P})$ be fixed where $P = P^{ij}_a \cup P^{jk}_b \cup P^{ik}_c$, $1 \leq i < j < k \leq t$, $1 \leq a, b, c \leq \ell$. With $\mathcal{G}_P = \mathcal{G} \cap \mathcal{K}^{(2)}_3(P)$, we set $\alpha_P = d_{\mathcal{G}_P}(P)$ as the density. It was shown in [5] that, with $\varepsilon$ sufficiently small,

$$\left|\mathcal{K}^{(3)}_{2,2,2}(\mathcal{G}_P)\right| \geq \frac{\alpha_P^8}{\ell^{12}} \binom{n}{2}^3 \left(1 - \varepsilon^{1/10}\right).$$

The following definition is therefore motivated.

**Definition 3 ($(\alpha, \delta)$-minimality)** *For $\delta > 0$, we say the 3-graph $\mathcal{G}_P$, as above, is* $(\alpha_P, \delta)$-minimal with respect to $P$ *if*

$$\left|\mathcal{K}^{(3)}_{2,2,2}(\mathcal{G}_P)\right| \leq \frac{\alpha_P^8}{\ell^{12}} \binom{n}{2}^3 (1 + \delta).$$

*If $\mathcal{G}_P$ is not $(\alpha_P, \delta)$-minimal with respect to $P$, then we say $\mathcal{G}_P$ is* $(\alpha_P, \delta)$-excessive *with respect to $P$.*

We may now state our algorithmic 3G-Lemma.

**Theorem 4** *For all $\delta > 0$ and $\alpha_0 > 0$, integers $t_0$ and $\ell_0$ and functions $\boldsymbol{\varepsilon} : \mathbb{N}^+ \to (0, 1)$, there exist integers $T_0$, $L_0$ and $N_0$ so that for every 3-graph $\mathcal{G}$ on vertex set $V$, $|V| = N > N_0$, there exist integers $\ell_0 \leq \ell \leq L_0$ and $t_0 \leq t \leq T_0$ and an $(\ell, t, \boldsymbol{\varepsilon}(\ell))$-partition $\mathcal{P}$ of $V$ such that the following holds.*

*For all but $\delta t^3 \ell^3$ triads $P \in \mathrm{Triad}(\mathcal{P})$ of density $d_{\mathcal{G}_P}(P) = \alpha_P \geq \alpha_0$, we have that $P$ is $(\alpha_P, \delta)$-minimal with respect to $\mathcal{G}_P$.*

*Moreover, there exists an algorithm which produces the partition $\mathcal{P}$ in time $O(N^6)$.*

### 3.1. Counting

Many applications of Szemerédi's Regularity Lemma are based on the fact that one can embed constant-sized subgraphs within an appropriately given $\varepsilon$-regular partition rendered by the Regularity Lemma. This result is formally due to the following "counting result", which follows from the definition of $\varepsilon$-regularity of graphs.

**Fact 5** *For all integers $k$ and non-negative $d$, there exists $\varepsilon_0 = \varepsilon_0(k,d) > 0$ so that for all $0 < \varepsilon < \varepsilon_0$, there exists integer $n_0$ so that whenever $H = \bigcup_{1\le i<j\le k} H^{ij}$ is a $k$-partite graph on $V_1 \cup \ldots \cup V_k$, $|V_1| = \ldots = |V_k| = n > n_0$, where each $H^{ij}$, $1 \le i < j \le k$, is $(d,\varepsilon)$-regular, then the number of $k$-cliques in $H$, $|\mathcal{K}_k^{(2)}(H)|$, satisfies $|\mathcal{K}_k^{(2)}(H)| = d^{\binom{k}{2}} n^k \left(1 \pm \varepsilon^{1/k}\right)$.*

Here the notation $a = b(1 \pm c)$, $a,b,c \ge 0$, means $b(1-c) \le a \le b(1+c)$. We note that the error term $\varepsilon^{1/k}$ in Fact 5 is not optimal and we use it here only because its form is convenient and will be consistent with other error estimates we make later.

We prove an analogous hypergraph Counting Lemma compatible with our algorithmic regularity lemma, Theorem 4. In what follows, for a hypergraph $\mathcal{G}$ on vertex set $V$ and an integer $k$, let $\mathcal{K}_k^{(3)}(\mathcal{G})$ denote the system of $k$-cliques in $\mathcal{G}$:

$$\mathcal{K}_k^{(3)}(\mathcal{G}) = \Big\{ K \in \binom{V}{k} : \; K \text{ induces a clique } K_k^{(3)} \text{ of size } k \text{ in } \mathcal{G} \Big\}.$$

The Counting Lemma is given as follows.

**Theorem 6 (Counting Lemma)** *For all integers $k$ and reals $\alpha_0 > 0$, there exists $\delta_0 = \delta_0(k,\alpha_0) > 0$ so that for all $0 < \delta < \delta_0$ and for all integers $\ell$ there exists $\varepsilon > 0$ and integer $n_0$ so that the following holds. Let $P = \bigcup_{1\le i<j\le k} P^{ij}$ be a $k$-partite graph with vertex classes $V = V_1 \cup \cdots \cup V_k$, $n_0 \le n \le |V_1| \le \cdots \le |V_k| \le n+1$, and let $\mathcal{H} = \bigcup_{1\le h<i<j\le k} \mathcal{H}^{hij}$ be a 3-graph where $\mathcal{H} \subseteq \mathcal{K}_3^{(2)}(P)$. Suppose that*

1. *for each $1 \le i < j \le k$, $P^{ij}$ is $(1/\ell, \varepsilon)$-regular,*
2. *for each $1 \le h < i < j \le k$, $\mathcal{H}^{hij}$ is $(\alpha_{hij}, \delta)$-minimal with respect to $P^{hi} \cup P^{ij} \cup P^{hj}$ where $\alpha_{hij} \ge \alpha_0$,*

*then*

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| = \frac{\prod_{1\le h<i<j\le k} \alpha_{hij}}{\ell^{\binom{k}{2}}} n^k \left(1 \pm \delta^{1/k}\right).$$

We make an important remark about the constants in Theorem 6. Observe that the quantification there renders the hierarchy

$$\frac{1}{k}, \alpha_0 \gg \delta \ge \min\left\{\delta, \frac{1}{\ell}\right\} \gg \varepsilon \gg \frac{1}{n}$$

which includes the case

$$\frac{1}{k}, \alpha_0 \gg \delta \gg \frac{1}{\ell} \gg \varepsilon \gg \frac{1}{n}. \qquad (2)$$

An application of the Regularity Lemma, Theorem 4, can't avoid the outcome (2), but it can enforce it. The quantification of Theorem 6 is therefore formulated as such so that Theorems 4 and 6 will be compatible.

## 4. Proof Sketches

In this section, we highlight important features of the proofs of Theorem 4 and 6. We begin with the former.

### 4.1. On Theorem 4

We require some background considerations and begin by defining the concept of $(\delta, r)$-regularity from [11]. Suppose $\mathcal{G}$ has vertex set $V$ and let $\mathcal{P}$ be an $(\ell, t, \varepsilon)$-partition of $V$. Fix a triad $P \in \mathrm{Triad}(\mathcal{P})$ and recall $\mathcal{G}_P = \mathcal{G} \cap \mathcal{K}_3^{(2)}(P)$.

**Definition 7 ($(\delta, r)$-regularity)** *For $\delta > 0$, and an integer $r$, we say $\mathcal{G}_P$ is* $(\delta, r)$-regular *w.r.t. $P$ if for any sequence $\boldsymbol{Q}_r$ of subgraphs $Q_i \subseteq P$, $1 \le i \le r$,*

$$\Big|\bigcup_{1\le i\le r} \mathcal{K}_3^{(2)}(Q_i)\Big| > \delta \Big|\mathcal{K}_3^{(2)}(P)\Big| \implies |d_{\mathcal{G}_P}(\mathbf{Q}_r) - d_{\mathcal{G}}(P)| < \delta$$

*where*

$$d_{\mathcal{G}_P}(\boldsymbol{Q}_r) = \frac{\left|\mathcal{G}_P \cap \bigcup_{1\le i\le r} \mathcal{K}_3^{(2)}(Q_i)\right|}{\left|\bigcup_{1\le i\le r} \mathcal{K}_3^{(2)}(Q_i)\right|}$$

*is the* density *of $\mathcal{G}_P$ w.r.t. the sequence $\boldsymbol{Q}_r$.*

If $\mathcal{G}_P$ is not $(\delta, r)$-regular w.r.t. $P$, then we say $\mathcal{G}_P$ is $(\delta, r)$-*irregular* w.r.t. $P$, and any sequence $\boldsymbol{Q}_r$ violating the regularity condition above is said to be a *witness* of the $(\delta, r)$-irregularity of $\mathcal{G}_P$ w.r.t. $P$.

For a 3-graph $\mathcal{G}$ on vertex set $V$ with $(\ell, t, \varepsilon)$-partition $\mathcal{P}$, define the *index* of $\mathcal{P}$ w.r.t. $\mathcal{G}$ as

$$\mathrm{ind}\,\mathcal{P} = \sum_{P \in \mathrm{Triad}(\mathcal{P})} d^2_{\mathcal{G}_P}(P) \Big|\mathcal{K}_3^{(2)}(P)\Big|.$$

The parameter $\operatorname{ind} \mathcal{P}$ was introduced by Szemerédi [37, 38] and is defined here according to [11]. It is an important but easy observation that $\operatorname{ind} \mathcal{P} \leq 1$ for any partition $\mathcal{P}$.

The following lemma (cf. Lemma 3.1, [19]) is the central idea in proving Theorem 4.

**Lemma 8** *For all $\alpha_0, \delta_B > 0$, there exists $\delta_A > 0$ so that for all integers $\ell$, there exist $\varepsilon = \varepsilon(\ell) > 0$ and an integer $r = r(\ell)$ so that the following holds for all sufficiently large $N$. Let $\mathcal{G}$ be a 3-graph with vertex set $V$ and $(\ell, t, \varepsilon)$-partition $\mathcal{P}$, where $|V| = N$. Let $P \in \operatorname{Triad}(\mathcal{P})$ satisfy*

1. *$d_{\mathcal{G}_P}(P) = \alpha_P \geq \alpha_0$,*
2. *$\mathcal{G}_P$ is $(\alpha_P, \delta_B)$-excessive w.r.t. $P$, i.e., $|\mathcal{K}^{(3)}_{2,2,2}(\mathcal{G}_P)| > (\alpha_P^8/\ell^{12})\binom{n}{2}^3(1+\delta_B)$.*

*Then, $\mathcal{G}_P$ is $(\delta_A, r)$-irregular w.r.t. $P$. Moreover, there exists an algorithm which, in time $O(N^5)$ constructs a witness $\boldsymbol{Q}_{r,P}$ of the $(\delta_A, r)$-irregularity of $\mathcal{G}_P$ w.r.t. $P$.*

Lemma 8 is an extension of an earlier result of the present authors and Dementieva (cf. Lemma 3.3, pp 299, [7]).

With Lemma 8, the proof of Theorem 4 is essentially at hand. Indeed, with appropriately defined constants, suppose 3-graph $\mathcal{G}$ with vertex set $V$, $|V| = N$, is given with $(\ell_{\rm old}, t_{\rm old}, \varepsilon(\ell_{\rm old}))$-partition $\mathcal{P}_{\rm old}$. We may easily check, in time $N^6$, if $\mathcal{G}_P$ is $(\alpha_P, \delta_B)$-minimal w.r.t. $P$. Indeed, for each of the $\binom{t}{3}\ell^3$ triads $P \in \operatorname{Triad}(\mathcal{P}_{\rm old})$, one counts $|\mathcal{K}^{(3)}_{2,2,2}(\mathcal{G}_P)|$. If fewer than $\delta_B t^3\ell^3$ triads $\mathcal{P} \in \operatorname{Triad}(\mathcal{P}_{\rm old})$ of density $d_{\mathcal{G}_P}(P) = \alpha_P \geq \alpha_0$ are $(\alpha_P, \delta_B)$-excessive, then $\mathcal{P}_{\rm old}$ is the partition promised by Theorem 4.

Otherwise, at least $\delta_B t^3 \ell^3$ triads $\mathcal{P} \in \operatorname{Triad}(\mathcal{P}_{\rm old})$ are $(\alpha_P, \delta_B)$-excessive. By Lemma 8, we therefore have 'many' $(\delta_A, r(\ell_{\rm old}))$-irregular triads $P \in \operatorname{Triad}(\mathcal{P}_{\rm old})$ in our partition, and we now explicitly resume the line of argument from [11]. Indeed, we use Lemmas 3.9 and 3.10, pp 145-149, from [11], to infer there exists a $(\ell_{\rm new}, t_{\rm new}, \varepsilon(\ell_{\rm new}))$-partition $\mathcal{P}_{\rm new}$ of $V$ (a 'refinement' of $\mathcal{P}_{\rm old}$) for which

$$\operatorname{ind} \mathcal{P}_{\rm new} \geq \operatorname{ind} \mathcal{P}_{\rm old} + \frac{\delta_A^4}{2}$$

holds and for which $\ell_{\rm new}$ and $t_{\rm new}$ are bounded by constants *independent* of the integer $N$. Thus this step could be repeated at most $2/\delta_A^4$ times before a partition satisfying Theorem 4 is reached.

What remains is to understand the complexity of the operation above. Lemma 8 constructs, in time $O(N^5)$, witnesses $\boldsymbol{Q}_{r,P}$ for every $P \in \operatorname{Triad}(\mathcal{P}_{\rm old})$ for which $\mathcal{G}_P$ is $(\alpha_P, \delta_B)$-excessive. With a little work, one may show from the methods in [11] (cf. Lemmas 3.9 and 3.10, pp 145–149, [11]) that $\mathcal{P}_{\rm new}$ can be constructed from these witnesses in time $O(N^2)$. Thus the complexity of the whole algorithm is determined by the $O(N^6)$ operation of counting $|\mathcal{K}^{(3)}_{2,2,2}(\mathcal{G}_P)|$ for each $P$, of which there are a constant number.

## 4.2. On Theorem 6

In the discussion that follows, we focus on the case of the Counting Lemma where all triad densities $\alpha_{hij}$, $1 \leq h < i < j \leq k$, are the same, i.e., $\alpha = \alpha_{hij}$, $1 \leq h < i < j \leq k$. The more general case formulated in Theorem 6 is, in fact, a corollary of this special case.

The proof of Theorem 6 proceeds by induction on $k \geq 3$, where the base case is trivial. With appropriate constants $k, \alpha, \delta, \ell, \varepsilon, n$ (cf. (2)), let $\mathcal{H}$ and $P$ be given as in the hypothesis of Theorem 6. To establish the inductive step, observe

$$\left|\mathcal{K}^{(3)}_k(\mathcal{H})\right| = \sum_{x \in V_1} \left|\mathcal{K}^{(2)}_{k-1}(L_x) \cap \mathcal{K}^{(3)}_{k-1}(\mathcal{H}[V_2, \ldots, V_k])\right| \tag{3}$$

where, for $x \in V_1$ fixed,

$$L_x = \Big\{\{y, z\} \in P : \{x, y, z\} \in \mathcal{H}\Big\} \tag{4}$$

is the *link graph* of $x$ and, as usual, $\mathcal{H}[V_2, \ldots, V_k]$ is the $(k-1)$-partite subsystem of $\mathcal{H}$ induced on the vertices $V_2 \cup \cdots \cup V_k$. While we discuss how to handle the intersection momentarily, we see (3) determines $|\mathcal{K}^{(3)}_k(\mathcal{H})|$ as a function of $|\mathcal{K}^{(3)}_{k-1}(\mathcal{H}[V_2, \ldots, V_k])|$ (which we handle with induction) and $|\mathcal{K}^{(2)}_{k-1}(L_x)|$ (which, importantly, is only a *graph* parameter).

We find it convenient to rewrite (3) in terms of the following bipartite graphs $\Lambda \subseteq \Pi$ with vertex bipartition $X \cup Y$:

1. set $X = V_1$ and $Y = \mathcal{K}^{(2)}_{k-1}(P[V_2, \ldots, V_k])$ so that $Y$ is the family of cliques $K^{(2)}_{k-1}$ in the $(k-1)$-partite graph $P[V_2, \ldots, V_k]$;
2. for $x \in X$ and $K^- \in Y$, let $\{x, K^-\} \in \Pi$ if, and only if, $K^- \subseteq N_P(x)$ where $N_P(x)$ is the neighborhood of $x$ in the graph $P$;
3. for $x \in X$ and $K^- \in Y$, let $\{x, K^-\} \in \Lambda$ if, and only if, $\binom{K^-}{2} \subseteq L_x$, where $\binom{K^-}{2} \subseteq P$ is the set of (graph) edges of the $(k-1)$-clique $K^- = K^{(2)}_{k-1}$.

Set $Y_{\mathcal{H}} = \mathcal{K}^{(3)}_{k-1}(\mathcal{H}[V_2, \ldots, V_k])$ and observe that $Y_{\mathcal{H}} \subseteq Y$. The crucial observation is that (3) may be reformulated as

$$\left|\mathcal{K}^{(3)}_k(\mathcal{H})\right| = \Big|\Lambda[X, Y_{\mathcal{H}}]\Big| = \sum_{K^- \in Y_{\mathcal{H}}} \deg_\Lambda(K^-). \tag{5}$$

Since our induction hypothesis gives $|Y_{\mathcal{H}}| = (\alpha^{\binom{k-1}{3}}/\ell^{\binom{k-1}{2}})n^{k-1}(1 \pm \delta^{1/(k-1)})$, it suffices to prove the following claim (the proof of which we discuss momentarily).

**Claim 9** *All but $3\delta^{1/(25(k-1))}|Y_{\mathcal{H}}|$ vertices $K^- \in Y_{\mathcal{H}}$ have degree $\deg_\Lambda(K^-) = (\alpha^{\binom{k-1}{2}}/\ell^{k-1})n(1 \pm 3\delta^{1/(24(k-1))})$.*

The lower bound for the Counting Lemma is now immediate:

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \geq \frac{\alpha^{\binom{k-1}{2}}}{\ell^{k-1}} n \left(1 - 3\delta^{1/(24(k-1))}\right)$$
$$\times \left(1 - 3\delta^{1/25((k-1))}\right) |Y_{\mathcal{H}}| \overset{(2)}{\geq} \frac{\alpha^{\binom{k}{3}}}{\ell^{\binom{k}{2}}} n^k \left(1 - \delta^{1/k}\right)$$

where the last inequality follows from our induction hypothesis. Proving the upper bound is essentially the same, provided we care for one additional detail. Observe from our hierarchy (2), i.e., $\delta \gg 1/\ell$, that even $3\delta^{1/(25(k-1))}|Y_{\mathcal{H}}|$ vertices $K^- \in Y_{\mathcal{H}}$ of large degree, say, for a simple example, $\deg_\Lambda(K^-) = n$, would ruin the formula we seek to prove:

$$\delta^{1/(25(k-1))} \frac{n^k}{\ell^{\binom{k-1}{2}}} \gg \frac{\alpha^{\binom{k}{3}}}{\ell^{\binom{k}{2}}} n^k.$$

To overcome this technicality, recall that $\Lambda \subseteq \Pi$ so that $\deg_\Lambda(K^-) \leq \deg_\Pi(K^-)$ for each $K^- \in Y$. It is standard to prove the graph $\Pi$ is $(\ell^{1-k}, \varepsilon^{1/k^3})$-regular, and is therefore very tightly controlled by the parameter $\varepsilon$ (cf. (2)). Standard arguments then finish the proof of the upper bound for Theorem 6.

Claim 9 asserts that vertices $K^- \in Y_{\mathcal{H}}$ are (virtually) degree regular in the graph $\Lambda$. This feature holds, in fact, over vertices $K^- \in Y$ (from which we then infer Claim 9). To prove (virtual) degree-regularity of vertices $K^- \in Y$, it suffices to prove the following stronger assertion.

**Claim 10** *For all but $\delta^{1/7}n$ vertices $x \in X$,*

$$\deg_\Lambda(x) = \left(\frac{\alpha}{\ell}\right)^{\binom{k-1}{2}} \left(\frac{n}{\ell}\right)^{k-1} \left(1 \pm \delta^{1/(7(k-1))}\right)$$

*and for all but $\delta^{1/7}n^2$ pairs $\{x, y\} \in \binom{X}{2}$,*

$$\deg_\Lambda(x, y) = \left(\frac{\alpha^2}{\ell}\right)^{\binom{k-1}{2}} \left(\frac{n}{\ell^2}\right)^{k-1} \left(1 \pm \delta^{1/(7(k-1))}\right).$$

In other words, setting $d = \alpha^{\binom{k-1}{2}}/\ell^{k-1}$ and $m = n^{k-1}/\ell^{\binom{k-1}{2}}$, Claim 10 asserts that most vertices $x \in X$ have degree essentially $dm$ and most pairs of vertices $\{x, y\} \in \binom{X}{2}$ have codegree essentially $d^2m$. Claim 9 then follows from Claim 10 by a standard Cauchy-Schwarz argument. The connection between Claims 9 and 10 is, in essence, analogous to Lemma 3.2 in [2].

To prove Claim 10, observe that for $x \in X$ fixed,

$$\deg_\Lambda(x) = \left|K_{k-1}^{(2)}(L_x)\right|$$

where $L_x$ is the link graph defined in (4). Similarly, for $x, y \in \binom{X}{2}$ fixed, set $L_{xy} = L_x \cap L_y$ to be the *colink* graph so that

$$\deg_\Lambda(x, y) = \left|K_{k-1}^{(2)}(L_{xy})\right|.$$

Using our hypothesis that $\mathcal{H}^{1ij}$, $1 < i < j \leq k$, is $(\alpha, \delta)$-minimal, it is not difficult to show (using standard Cauchy-Schwarz and double-counting arguments) that most links $L_x$, $x \in X$, and most colinks $L_{xy}$, $\{x, y\} \in \binom{X}{2}$, satisfy the hypothesis of the following auxiliary graph counting lemma, from which both estimates of Claim 10 then follow.

**Lemma 11** *For all integers $t$ and constants $\lambda_0 \geq 0$ there exists $\delta_{0,\lambda_0} > 0$ so that for all $0 < \delta_{\lambda_0} \leq \delta_{0,\lambda_0}$ and $p > 0$ there exists $\varepsilon_p > 0$ and integer $m_0$ so that the following holds:*

*Let*

$$\mathfrak{L} = \bigcup_{1 \leq i < j \leq t} \mathfrak{L}^{ij} \subseteq \mathfrak{P} = \bigcup_{1 \leq i < j \leq t} \mathfrak{P}^{ij}$$

*be $t$-partite graphs with common $t$-partition $U_1 \cup \ldots \cup U_t$, $|U_i| = m > m_0$, satisfying that for all $1 \leq i \neq j \leq t$,*

1. *$|\mathfrak{L}^{ij}|/|\mathfrak{P}^{ij}| = \lambda \geq \lambda_0$,*
2. *all but $\delta_{\lambda_0} m^2$ pairs $u_1, u_2 \in U_i$ satisfy*
$$\deg_{\mathfrak{L}^{ij}}(u_1, u_2) = (\lambda p)^2\, m\, (1 \pm \delta_{\lambda_0})\,, \qquad (6)$$
3. *the graph $\mathfrak{P}^{ij}$ is $(p, \varepsilon_p)$-regular.*

*Then,*

$$\left|\mathcal{K}_t^{(2)}(\mathfrak{L})\right| = (\lambda p)^{\binom{t}{2}} m^t \left(1 \pm \delta_{\lambda_0}^{1/t}\right).$$

Indeed, one can prove that Lemma 11 applies to most links $L_x$, $x \in V_1$, with parameters $t = k-1$, $\lambda = \alpha$, $p = 1/\ell$ and $m = n/\ell$ and to most colinks $L_{xy}$, $\{x, y\} \in \binom{V_1}{2}$, with parameters $t = k-1$, $\lambda = \alpha^2$, $p = 1/\ell$ and $m = n/\ell^2$.

Proving Lemma 11 is not difficult and follows lines similar to (but simpler than) our approach for proving Theorem 6. Again we construct auxiliary bipartite graphs $\mathbb{L} \subseteq \mathbb{P}$ with bipartition $A \cup B$ as follows:

1. $A = U_1$ and $B = \mathcal{K}_{t-1}^{(2)}(\mathfrak{P}[U_2, \ldots, U_t])$;
2. for $a \in A$ and $K^- \in B$, let $\{a, K^-\} \in \mathbb{P}$ if, and only if, $K^- \subseteq N_{\mathfrak{P}}(x)$;
3. for $a \in A$ and $K^- \in B$, let $\{a, K^-\} \in \mathbb{L}$ if, and only if, $K^- \subseteq N_{\mathfrak{L}}(x)$.

Convexity, degree and co-degree regularity arguments similar to Claims 9 and 10 establish Lemma 11.

## 5. Proof of Theorem 1

We sketch the proof of Theorem 1 in the special case $\mathcal{J}_0 = K_k^{(3)}$ (in which case, $\mathcal{J}_0(\mathcal{G}) = \mathcal{J}_0^{\mathrm{ind}}(\mathcal{G})$), although the argument isn't much different for general $\mathcal{J}_0$.

Our basic approach is to apply Theorem 4 to $\mathcal{G}$ with appropriately chosen parameters, to obtain a partition $\mathcal{P}$. Then with very few exceptions (we will show how to count these), all copies of $K_k^{(3)}$ lie in $k$-partite 3-graphs satisfying the assumptions of Theorem 6. Moreover, the number of these $k$-partite 3-graphs depends only on the input parameters $k$ and $\eta$, and not on $N$. Thus in each $k$-partite 3-graph we may apply Theorem 6 to find the number of copies of $K_k^{(3)}$ in each, then sum them all to find a close approximation to $|\mathcal{K}_k^{(3)}(\mathcal{G})|$.

We now make these steps more precise. To choose the input parameters to Theorem 4 we do the following. Given $k$ and $\eta$, we choose $\delta$ and the function $\varepsilon(\ell)$ such that

- $\delta < \min\{(\eta/4)^k, \delta_0(k, \eta/4)\}$ (cf. Theorem 6),
- $\varepsilon(\ell) < \varepsilon_0(3, \ell^{-1})$ (cf. Fact 5),
- Theorem 6 holds for $k$, $\alpha_0 = \eta/4$ and $\delta$.

Then the input parameters for Theorem 4 are $\delta$, $\alpha_0 = \eta/4$, $t_0 = 100/\eta$, $\ell_0 = 1$, and function $\varepsilon(\ell)$. Theorem 4 provides constants $T_0$, $L_0$ and $N_0$, and a suitable partition $\mathcal{P}$ of the vertex set $V$ of $\mathcal{G}$, in time $O(N^6)$, where the constant hidden in the $O$ notation depends only on the input parameters and hence only on $k$ and $\eta$.

Now note that any triple $e$ of $\mathcal{G}$ that does not lie in an $(\alpha_P, \delta)$-minimal triad $P$ of $\mathcal{P}$ of density $\alpha_P > \alpha_0 = \eta/4$ satisfies one of the following:

- $e$ has two vertices inside the same $V_i$. There are at most $tn^2N < N^3/t_0 \le \eta N^3/100$ such triples,
- $e$ lies in a triad $P$ that is $(\alpha_P, \delta)$-excessive, of which there are at most $\delta t^3 \ell^3$. Because $\varepsilon(\ell) < \varepsilon_0(3, \ell^{-1})$, by Fact 5 each $P$ has at most $\ell^{-3}n^3(1 + \varepsilon(\ell)^{1/3})$ triangles, so the number of such triples is at most $\delta t^3 \ell^3(\ell^{-3}n^3(1 + \varepsilon(\ell)^{1/3})) < 2\delta t^3 n^3 \le 2\delta N^3 < 2(\eta/4)^k N^3 < \eta N^3/32$,
- $e$ lies in a triad $P$ that has density $\alpha_P < \alpha_0$. Thus by definition of density there can certainly be at most $\alpha_0|\mathcal{K}_3^{(2)}(K_N)|$ such triples, where $K_N$ is the complete graph on the vertex set $V$. Therefore there are at most $\alpha_0\binom{N}{3} < \eta N^3/24$ such triples.

We call these triples *bad* for $\alpha_0 = \eta/4$.

We therefore see that, given $k$ and $\eta$, if $N > N_0$, we may conclude that the number of bad triples for $\eta/4$ in $\mathcal{G}$ is less than $\eta N^3/4$. Thus the number of copies of $K_k^{(3)}$ in $\mathcal{G}$ that contain a bad triple is less than $\eta N^k/4$. Thus to estimate $|\mathcal{K}_k^{(3)}(\mathcal{G})|$ we simply count the copies of $K_k^{(3)}$ in $\mathcal{G}$ that do not contain a bad triple.

In $\mathcal{P}$ there are at most $t^k \ell^{\binom{k}{2}} < T_0^k L_0^{\binom{k}{2}}$ $k$-partite 3-graphs $\mathcal{H}$ satisfying the assumptions of Theorem 6 with $\alpha_0 = \eta/4$. The key point here is that the number of such $\mathcal{H}$ is independent of $N$. Each copy of $K_k^{(3)}$ in $\mathcal{G}$ whose triples are all good for $\eta/4$ lies in one of these $\mathcal{H}$. For each $\mathcal{H}$, Theorem 6 tells us how many copies of $K_k^{(3)}$ it contains, up to a multiplicative factor of $(1 \pm \eta/4)$. Thus counting over all $\mathcal{H}$ gives an estimate of $|\mathcal{K}_k^{(3)}(\mathcal{G})|$ that differs from the exact value by at most $\eta N^k$. Because the number of such $\mathcal{H}$ is a constant independent of $N$, the complexity of the whole algorithm is still $O(N^6)$.

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