ON EXTENDING HANSEL'S THEOREM TO HYPERGRAPHS

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Abstract. For integers $2 \leq d \leq k \leq n$, a *d-covering* of $\binom{[n]}{k}$ is a family $\mathcal{D}$ of d-partite k-graphs D, where every k-tuple of $\binom{[n]}{k}$ belongs to some $D \in \mathcal{D}$. When $d = k$, a k-covering is called a *covering* of $\binom{[n]}{k}$. Determining the minimum size of a covering of $\binom{[n]}{k}$ has been studied by many authors, where the best bounds are due to Fredman and Komlós [3]. In this note, we consider a variation of this problem for d-coverings, when $d \leq k$.

Let $f(n, k, d)$ denote the minimum of $\sum_{D \in \mathcal{D}} |V(D)|$, over all d-coverings of $\binom{[n]}{k}$. When $d = k = 2$, a classical theorem of Hansel says that $\lceil n \log_2 n \rceil \leq f(n, 2, 2) \leq n \lceil \log_2 n \rceil$. Our first result gives a partial generalization of Hansel's theorem, for all integers $2 \leq k \leq n$:

$$\left\lceil n \log_2 \left(\frac{n}{k-1} \right) \right\rceil \leq f(n, k, 2) \leq n \left\lceil \log_2 \left\lceil \frac{n}{k-1} \right\rceil \right\rceil .$$

Our second result shows that, for $2 \leq d \leq k \leq n$, we have $f(n, k, d) \geq n \log_{d/(d-1)}(n/(k-1))$. Our third result shows that this lower bound is asymptotically best possible for a non-trivial range of $2 \leq d \leq k \leq n$.

1. Introduction

In this paper, we always work with vertices within $[n] = \{1, \ldots, n\}$. Let $\binom{[n]}{k}$ be the collection of all k-element subsets of $[n]$. A *k-uniform hypergraph* H, or *k-graph* for short, is any subcollection $H \subseteq \binom{[n]}{k}$, the elements of which we call *edges*. We denote by $V(H) \subseteq [n]$ the set of vertices which belong to edges of H. We say that H is *k-partite* if there exists a partition $V(H) = V_1 \cup \cdots \cup V_k$ so that each edge of H meets each V_i, $1 \leq i \leq k$. (Note that, in this case, each edge of H will meet each such V_i in precisely one vertex.) A *covering* of $\binom{[n]}{k}$ is a family $\mathcal{H}$ of k-partite k-graphs H, where every k-tuple of $\binom{[n]}{k}$ is an edge of some $H \in \mathcal{H}$.

The minimum size $|\mathcal{H}|$ over all coverings $\mathcal{H}$ of $\binom{[n]}{k}$ has applications to perfect hashing in theoretical computer science, and has been studied by many authors (see, e.g., [3, 7, 9, 11, 12]). The best known lower bound on $|\mathcal{H}|$ is due to Fredman and Komlós [3] (cf. [7, 11]), and is given by $|\mathcal{H}| \geq (1 - o(1))(e^k/(k\sqrt{2\pi k})) \log n$, where $o(1) \to 0$ as $n \to \infty$. (An alternative argument for this bound was later found by Nilli [9].) The bound of Fredman and Komlós is not far from the best known upper bound $O(\sqrt{k} e^k \log n)$, which follows from a standard application of the probabilistic method.

When $k = 2$, let $f(n)$ denote the minimum of $\sum_{H \in \mathcal{H}} |V(H)|$ over all coverings $\mathcal{H}$ of $\binom{[n]}{2}$. A classical result of Hansel [4] says the following.

Theorem 1.1 (Hansel [4])**.** $\lceil n \log_2 n \rceil \leq f(n) \leq n \lceil \log_2 n \rceil$.

Hansel's theorem was proven in the context of computing Boolean functions, and a similar result was independently and concurrently proven by Krichevskii [8]. Katona and Szemerédi [6] rediscovered Hansel's theorem in the context of a diameter problem in graph theory. More recently,

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Bollobás and Scott [1] sharpened Hansel's theorem to a precise formula for the parameter $f(n)$, for all integers n (see upcoming Remark 1.5).

In this note, we consider a partial extension of Hansel's theorem to hypergraphs. For integers $2 \leq d \leq k \leq n$, we say that a k-graph $D \subseteq \binom{[n]}{k}$ is *d-partite* if there exists a partition $V(D) = V_1 \cup \cdots \cup V_d$ so that every edge of D meets each V_i, $1 \leq i \leq d$, in at least one vertex. A *d-covering* of $\binom{[n]}{k}$ is a family $\mathcal{D}$ of d-partite k-graphs D, where every k-tuple of $\binom{[n]}{k}$ is an edge of some $D \in \mathcal{D}$. Let $f(n, k, d)$ denote the minimum of $\sum_{D \in \mathcal{D}} |V(D)|$ over all d-coverings of $\binom{[n]}{k}$. Our first result concerns the case $d = 2$

Theorem 1.2. *For all integers* $2 \leq k \leq n$,

$$\left\lceil n \log_2 \left(\frac{n}{k-1} \right) \right\rceil \leq f(n, k, 2) \leq n \left\lceil \log_2 \left\lceil \frac{n}{k-1} \right\rceil \right\rceil .$$

Note that if $k = 2$, then Theorem 1.2 reduces to Theorem 1.1. Note also that if $n/(k-1)$ is an integer power of two, then Theorem 1.2 gives an exact formula for $f(n, k, 2)$ (see upcoming Remark 1.5).

The upper bound of Theorem 1.2 follows from an elementary argument, which we give in Section 2. For the lower bound of Theorem 1.2, we prove the following more general result. (This proof is given in Section 3.)

Theorem 1.3. *For all integers* $2 \leq d \leq k \leq n$,

$$f(n, k, d) \geq \left\lceil n \log_{d/(d-1)} \left(\frac{n}{k-1} \right) \right\rceil .$$

The proof of Theorem 1.3 employs elegant ideas from [4, 6, 9] (which are nicely presented in [5]). We use these ideas to demonstrate, in fact, a result on d-coverings of arbitrary k-graphs.

Theorem 1.2 shows that Theorem 1.3 is best possible when $d = 2$ and $n/(k-1)$ is an integer power of two. Since more information on $f(n, k, d)$ could be of interest, we note the following range of $2 \leq d \leq k \leq n$ where Theorem 1.3 is again relevant.

Theorem 1.4. *For a positive integer variable n, let $k = k(n) = O(\sqrt{\log \log n})$ be a diverging integer-valued function of n, and let $2 \leq d = d(k) = O(k/\log^2 k)$ be an integer-valued function of k (hence, $d = d(n)$ is a function of n). Then, for large integers n,*

$$f(n, k, d) = \left(1 + O\left(\frac{d \log d}{k} \right) \right) n \log_{d/(d-1)} \left(\frac{n}{k-1} \right) = (1 + o(1)) n \log_{d/(d-1)} \left(\frac{n}{k-1} \right) .$$

We prove Theorem 1.4 in Section 4, where we follow standard probabilistic techniques.

Remark 1.5. In the course of submitting this paper, we became able to prove an exact formula for the parameter $f(n, k, 2)$ for all integers $2 \leq k \leq n$. We address these details in a forthcoming paper [2].

2. The upper bound of Theorem 1.2

Fix integers $2 \leq k \leq n$. We construct a d-covering $\mathcal{D}$ of $\binom{[n]}{k}$ satisfying $\sum_{D \in \mathcal{D}} |V(D)| = n \lceil \log_2 \lceil n/(k-1) \rceil \rceil$. To that end, we first make a few auxiliary considerations. Set $p = \lfloor n/(k-1) \rfloor$ and $q = \lceil n/(k-1) \rceil$. Let $[n] = U_1 \cup \cdots \cup U_q$ be any partition (coloring) with $|U_1| = \cdots = |U_p| = k-1$, which is necessarily a proper vertex coloring of $\binom{[n]}{k}$ (since $|U_1|, \ldots, |U_q| < k$). Set $m = \lceil \log_2 q \rceil$, and let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_q \in \{0,1\}^m$ be distinct. Define the mapping $u \mapsto \boldsymbol{u} \in \{0,1\}^m$ by, for each $(u, i) \in [n] \times [q]$, $u \mapsto \boldsymbol{u}_i$ if, and only if, $u \in U_i$.

To define $\mathcal{D}$, fix $j \in [m]$, and set $A_j = \{u \in [n] : \boldsymbol{u}(j) = 0\}$ and $B_j = \{u \in [n] : \boldsymbol{u}(j) = 1\}$, where $\boldsymbol{u}(j)$ denotes the j^{th} coordinate of $\boldsymbol{u}$. Let $D_j = K^{(k)}[A_j, B_j]$ be the complete bipartite k-graph with vertex bipartition $V(D_j) = [n] = A_j \cup B_j$, and set $\mathcal{D} = \{D_1, \dots, D_m\}$. Clearly,

$$\sum_{D \in \mathcal{D}} |V(D)| = \sum_{j=1}^{m} (|A_j| + |B_j|) = nm = n\lceil \log_2 q \rceil = n \left\lceil \log_2 \left\lceil \frac{n}{k-1} \right\rceil \right\rceil .$$

It remains to verify that $\mathcal{D}$ is a 2-covering of $\binom{[n]}{k}$. To that end, fix a k-tuple $K \in \binom{[n]}{k}$. Since $U_1 \cup \cdots \cup U_q$ is a proper vertex coloring of $\binom{[n]}{k}$, let $u_h \in K \cap U_h$ and $u_i \in K \cap U_i$ hold for some $1 \le h < i \le q$. Then, $\boldsymbol{u}_h \ne \boldsymbol{u}_i$, and so there exists $j \in [m]$ so that, w.l.o.g., $\boldsymbol{u}_h(j) = 0$ and $\boldsymbol{u}_i(j) = 1$. In this case, $u_h \in A_j$ and $u_i \in B_j$, and hence $K \in K^{(k)}[A_j, B_j] = D_j$.

3. Proof of Theorem 1.3

We prove the following more general result. Fix integers $2 \le d \le k \le n$, and fix a k-graph $H \subseteq \binom{[n]}{k}$. A *d-covering* of H is a family $\mathcal{D}$ of d-partite k-graphs D, where every k-tuple of H is an edge of some $D \in \mathcal{D}$. An *independent set* in H is a set $A \subseteq [n]$ which spans no edges of H, i.e., the induced subhypergraph $H[A] = H \cap \binom{A}{k}$ is empty. As usual, let $\alpha(H)$ denote the maximum size of an independent set in H.

Theorem 3.1. *For integers $2 \le d \le k \le n$, and for a d-covering $\mathcal{D}$ of a k-graph $H \subseteq \binom{[n]}{k}$,*

$$\sum_{D \in \mathcal{D}} |V(D)| \ge \left\lceil n \log_{d/(d-1)} (n/\alpha(H)) \right\rceil .$$

Theorem 3.1 then implies Theorem 1.3, since the k-clique $H = \binom{[n]}{k}$ has $\alpha(H) = k-1$.

Proof of Theorem 3.1. Fix integers $2 \le d \le k \le n$, and let $\mathcal{D} = \{D_1, \dots, D_t\}$ be a d-covering of a k-graph $H \subseteq \binom{[n]}{k}$. We write, for each $i \in [t]$, the d-partition of D_i as $V_i = V(D_i) = V_{i,1} \cup \cdots \cup V_{i,d}$. We prove $\sum_{i=1}^{t} |V_i| \ge n \log_{d/(d-1)}(n/\alpha(H))$.

We consider the following random subset $W \subseteq [n]$: select $\boldsymbol{j} = (j_1, \dots, j_t) \in [d]^t$ uniformly at random; for each $i \in [t]$, set $W_i = [n] \setminus V_{i,j_i}$; set $W = \bigcap_{i=1}^{t} W_i$. Observe that W is an independent set in H. Indeed, if $K \in H[W]$, then there exists $i \in [t]$ so that $K \in D_i$, and so K meets each of $V_{i,1}, \dots, V_{i,d}$. On the other hand, $K \subseteq W \subseteq W_i = [n] \setminus V_{i,j_i}$, so that $K \cap V_{i,j_i} = \emptyset$, a contradiction. Note that the independence of W gives $|W| \le \alpha(H)$, and therefore

$$\mathbb{E}[|W|] \le \alpha(H). \tag{1}$$

We next develop an exact expression for $\mathbb{E}[|W|]$ (see (4) below).

Fix $v \in [n]$, and set $X_v = 1$ if $v \in W = \bigcap_{i=1}^{t} W_i$, and $X_v = 0$ otherwise. Note that, since $\boldsymbol{j} \in [d]^t$ is selected uniformly at random, the events $v \in W_i$, over $i \in [t]$, are independent. Then $|W| = \sum_{v \in [n]} X_v$, and by linearity of expectation,

$$\mathbb{E}[|W|] = \sum_{v \in [n]} \mathbb{E}[X_v] = \sum_{v \in [n]} \mathbb{P}\left[v \in \bigcap_{i=1}^{t} W_i \right] = \sum_{v \in [n]} \prod_{i=1}^{t} \mathbb{P}\left[v \in W_i \right] = \sum_{v \in [n]} \prod_{i=1}^{t} \mathbb{P}\left[v \notin V_{i,j_i} \right]. \tag{2}$$

For fixed $(v, i) \in [n] \times [t]$, observe that

$$\mathbb{P}\left[v \notin V_{i,j_i} \right] = \begin{cases} 1 & \text{if } v \notin V_i, \\ (d-1)/d & \text{if } v \in V_i. \end{cases} \tag{3}$$

Indeed, to avoid triviality, let $v \in V_i$, and let $j_v \in [d]$ be the unique index for which $v \in V_{i,j_v}$. Then, $\mathbb{P}[v \notin V_{i,j_i}] = \mathbb{P}[j_i \neq j_v] = 1 - \mathbb{P}[j_i = j_v] = 1 - (1/d)$, as promised in (3).

Returning to (2), define auxiliary family $\mathcal{F} = \{V_1, \ldots, V_t\}$, so that a fixed $v \in [n]$ belongs to precisely $\deg_{\mathcal{F}}(v)$ many elements $V_i \in \mathcal{F}$. As such,

$$\prod_{i=1}^{t} \mathbb{P}\left[v \notin V_{i,j_i}\right] \overset{(3)}{=} \left(\frac{d-1}{d}\right)^{\deg_{\mathcal{F}}(v)} \quad \overset{(2)}{\Longrightarrow} \quad \mathbb{E}[|W|] = \sum_{v \in [n]} \left(\frac{d-1}{d}\right)^{\deg_{\mathcal{F}}(v)}. \tag{4}$$

Comparing (1) and (4), and using the Arithmetic-Geometric mean inequality, we infer

$$\frac{\alpha(H)}{n} \geq \frac{1}{n}\mathbb{E}[|W|] = \frac{1}{n} \sum_{v \in [n]} \left(\frac{d-1}{d}\right)^{\deg_{\mathcal{F}}(v)}$$
$$\geq \left[\prod_{v \in [n]} \left(\frac{d-1}{d}\right)^{\deg_{\mathcal{F}}(v)}\right]^{1/n} = \left(\frac{d-1}{d}\right)^{(1/n)\sum_{v \in [n]} \deg_{\mathcal{F}}(v)}.$$

By standard double-counting, we have $\sum_{v \in [n]} \deg_{\mathcal{F}}(v) = \sum_{i=1}^{t} |V_i|$, from which it follows that $\sum_{i=1}^{t} |V_i| \geq n \log_{d/(d-1)}(n/\alpha(H))$. $\square$

4. Proof of Theorem 1.4

Let n, $k = k(n) = O(\sqrt{\log\log n})$, and $2 \leq d = d(k) = O(k/\log^2 k)$ be given as in Theorem 1.4. To bound $f(n,k,d)$, we use a standard random construction to produce a d-covering $\mathcal{D}$ of $\binom{[n]}{k}$ for which $\sum_{D \in \mathcal{D}} |V(D)|$ is not too large. To that end, define auxiliary positive integer parameter

$$m = \left\lceil -\frac{(k-1)\log(n/k)}{\log\left(d(1-(1/d))^k\right)} \right\rceil, \tag{5}$$

where for simplicity in calculations, we will ignore the ceilings. For a function $\phi : [n] \to [d]^m$, we write $\boldsymbol{v} = \phi(v) = (\boldsymbol{v}(1), \ldots, \boldsymbol{v}(m))$. For a fixed $K = \{v_1, \ldots, v_k\} \in \binom{[n]}{k}$, we say that K is *ϕ-separated* if, for some $i \in [m]$, we have $\{\boldsymbol{v}_1(i), \ldots, \boldsymbol{v}_k(i)\} = [d]$. Moreover, we define $X_{\phi,K}$ to be the indicator variable for when K is not ϕ-separated, and we set $X_\phi = \sum_{K \in \binom{[n]}{k}} X_{\phi,K}$.

Select $\varphi : [n] \to [d]^m$ uniformly at random. We will observe that for each $K \in \binom{[n]}{k}$,

$$\mathbb{E}[X_{\varphi,K}] = \mathbb{P}[K \text{ is not } \varphi\text{-separated}] \leq (k/n)^{k-1},$$
$$\text{in which case } \mathbb{E}[X_\varphi] = \sum_{K \in \binom{[n]}{k}} \mathbb{E}[X_{\varphi,K}] \leq \binom{n}{k}\left(\frac{k}{n}\right)^{k-1} \leq \left(\frac{en}{k}\right)^k \left(\frac{k}{n}\right)^{k-1} = e^k \frac{n}{k}. \tag{6}$$

Indeed, there are at least $d^k - d(d-1)^k$ many surjections $K \overset{\text{onto}}{\to} [d]$, and so

$$\mathbb{E}[X_{\varphi,K}] \leq \frac{\left(d^k - (d^k - d(d-1)^k)\right)^m \times d^{m(n-k)}}{d^{mn}} = \left[d\left(1 - \frac{1}{d}\right)^k\right]^m$$
$$= \exp\left\{m \log\left[d\left(1-\frac{1}{d}\right)^k\right]\right\} \overset{(5)}{=} \exp\left\{-(k-1)\log(n/k)\right\} = (k/n)^{k-1}.$$

We now define the promised family $\mathcal{D}$. Fix any $\phi : [n] \to [d]^m$ for which $X_\phi \leq \mathbb{E}[X_\varphi]$. For each $(i,j) \in [m] \times [d]$, set $V_{i,j} = \{v \in [n] : \boldsymbol{v}(i) = j\}$, and set $D_i = K^{(k)}[V_{i,1}, \ldots, V_{i,d}]$.

For each $K = \{v_1, \ldots, v_k\} \in \binom{[n]}{k}$, define $D_K = K^{(k)}[\{v_1\}, \ldots, \{v_{d-1}\}, \{v_d, \ldots, v_k\}]$. Define $\mathcal{D} = \mathcal{D}_\phi = \{D_1, \ldots, D_m\} \cup \{D_K : K \in \binom{[n]}{k}$ is not ϕ-separated$\}$. By construction, $\mathcal{D}$ is a d-covering of $\binom{[n]}{k}$, which satisfies

$$\sum_{D \in \mathcal{D}} |V(D)| = kX_\phi + \sum_{i=1}^{m} |V(D_i)| \leq k\mathbb{E}[X_\varphi] + mn \overset{(6)}{\leq} mn + e^k n = mn\left(1 + \frac{e^k}{m}\right).$$

We claim that

$$m = \left(1 + O\left(\frac{d \log d}{k}\right)\right) \log_{d/(d-1)}\left(\frac{n}{k-1}\right) \quad \text{and} \quad \frac{e^k}{m} = O\left(\frac{1}{k}\right), \tag{7}$$

which, if true, immediately implies Theorem 1.4. To see (7), note first that the denominator $-\log(d(1-(1/d))^k)$ in (5) equals

$$k \log\left(\frac{d}{d-1}\right)\left(1 + \frac{\log d}{k \log(1-(1/d))}\right) = k \log\left(\frac{d}{d-1}\right)\left(1 - \Theta\left(\frac{d \log d}{k}\right)\right), \tag{8}$$

where we used that $\log(1+x) \approx x$ for $x \approx 0$. Since $d(\log d)/k = o(1)$ holds by hypothesis,

$$m \overset{(5)}{=} \frac{(k-1)\log(n/k)}{-\log(d(1-(1/d))^k)} \leq \frac{k \log(n/(k-1))}{k \log(d/(d-1))(1 - \Theta(d(\log d)/k))}$$

satisfies (7), using $(1-x)^{-1} \leq 1+2x$ (on $[0, 1/2]$). Moreover, since (8) is $(1-o(1))k\log(d/(d-1))$, where $k = O(\sqrt{\log\log n})$ diverges, we have $m \geq (1-o(1))\log_{d/(d-1)} n \geq (1-o(1))\log n$, while $ke^k = O(e^{k^2}) = O(\log n) = O(m)$. Thus, e^k/m satisfies (7).

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