TURÁN RELATED PROBLEMS FOR HYPERGRAPHS

BRENDAN NAGLE

EMORY UNIVERSITY, ATLANTA, GA. 30032, USA
E-mail address: nagle@mathcs.emory.edu

ABSTRACT. For an l-uniform hypergraph $\mathcal{F}$ and an integer n, the Turán number $ex(n, \mathcal{F})$ for $\mathcal{F}$ on n vertices is defined to be the maximum size $|\mathcal{G}|$ of a hypergraph $\mathcal{G} \subseteq [n]^l$ not containing a copy of $\mathcal{F}$ as a subhypergraph. For $l = 2$ and $\mathcal{F} = K_k^{(2)}$, the complete graph on k vertices, these numbers were determined by P. Turán. However, for $l > 2$, and nearly any hypergraph $\mathcal{F}$, the Turán problem of determining the numbers $ex(n, \mathcal{F})$ has proved to be very difficult, and very little about these numbers is known. In this survey, we discuss recent results and open problems for triple systems which relate to Turán numbers $ex(n, \mathcal{F})$.

1. INTRODUCTION

In this survey, we discuss extremal problems for l-uniform hypergraphs which relate to a classical problem of Turán. In Section 1, we review the classical Turán problem. In Section 2, we discuss applications of a regularity lemma of [FR] that relate to the Turán problem. In Section 3, we discuss problems and results introduced by a question of Abassi that relate to the classical problem of Turán.

For a natural number l, and a finite set V, any $\mathcal{G} \subseteq [V]^l = \{B \subseteq V : |B| = l\}$ is called an *l-uniform hypergraph*. We identify an l-uniform hypergraph $\mathcal{G}$ as the set of l-tuples of the vertex set $V = V(\mathcal{G})$ of which it is comprised. If $l = 2$, we speak about *graphs*, and if $l = 3$, we speak about *triple systems*. As part of our basic notation in what follows, for an integer n, we will use the notation $[n]$ to denote the set $\{1, \dots, n\}$.

A concept fundamental to all problems discussed in this paper is that of *forbidden subhypergraphs*. For a fixed l-uniform hypergraph $\mathcal{F}$, we call an l-uniform hypergraph $\mathcal{G}$ an *$\mathcal{F}$-free hypergraph* if $\mathcal{G}$ has no subhypergraph isomorphic to $\mathcal{F}$ ($\mathcal{J}$ is a subhypergraph of $\mathcal{G}$ if $\mathcal{J} \subseteq \mathcal{G}$). For a fixed l-uniform hypergraph $\mathcal{F}$ and an integer n, we define the *Turán number for $\mathcal{F}$ on n vertices*, denoted by $ex(n, \mathcal{F})$, to be the largest size $|\mathcal{G}|$ of any $\mathcal{F}$-free $\mathcal{G} \subseteq [n]^l$. It is easy to show that for any fixed $\mathcal{F}$, $\lim_{n\to\infty} \frac{ex(n,\mathcal{F})}{\binom{n}{l}}$ exists. We thus set

$$ex(\mathcal{F}) = \lim_{n\to\infty} \frac{ex(n, \mathcal{F})}{\binom{n}{l}}$$

CONGRESSUS NUMERANTIUM **136** (1999), pp.119-127

and refer to this limit as the *Turán number for* $\mathcal{F}$.

Turán's original problem was the following.

Problem 1.1. (Turán) *For integers* k, l, $k \geq l > 2$, *and for the fixed* l*-uniform hypergraph* $\mathcal{F} = K_k^{(l)} = [k]^l$, *the clique on* k *vertices, determine*

$$ex(K_k^{(l)}) = \lim_{n\to\infty} \frac{ex(n, K_k^{(l)})}{\binom{n}{l}}$$

In the case of $l = 2$, Turán was able to find both the asymptotic requested above as well as a formula for $ex(n, K_k^{(2)})$, for all positive integers n.

Theorem 1.2. (Turán [1940]) *Let* n *and* k *be positive integers,* $n \geq k$, *and let* m *and* r *be nonnegative integers satisfying that* $n = mk + r$, *where* $0 \leq r \leq k-1$. *Then*

$$ex(n, K_{k+1}^{(2)}) = m^2\binom{k}{2} + rm(k-1) + \binom{r}{2}$$

Consequently,

$$ex(K_{k+1}^{(2)}) = 1 - \frac{1}{k}$$

Furthermore, Turán showed that all $K_{k+1}^{(2)}$-free $\mathcal{G} \subseteq [n]^2$ are isomorphic to the complete k-partite graph

$$K^{(2)}(V_1, \ldots, V_k) = \{\{v_i, v_j\} : v_i \in V_i,\ v_j \in V_j,\ 1 \leq i < j \leq k\},$$

where the partite sets $\{V_1, \ldots, V_k\}$ satisfy that for each $i \in [k]$,

$$\lfloor \frac{n}{k} \rfloor \leq |V_i| \leq \lceil \frac{n}{k} \rceil$$

Turán's problem is notoriously open. For the simplest nontrivial case when $l = 3$ and $k = 4$, Turán conjectured that these Turán numbers should be "nice". Specifically,

Conjecture 1.3. (Turán)

$$ex(n, K_4^{(3)}) = \begin{cases} \frac{m^2(5m-3)}{2} & \text{if } n = 3m \\ \frac{m(5m^2+2m-1)}{2} & \text{if } n = 3m+1 \\ \frac{m(m+1)(5m+2)}{2} & \text{if } n = 3m+2 \end{cases} \tag{1}$$

Consequently,

$$ex(K_4^{(3)}) = \frac{5}{9} \tag{2}$$

Conjecture 1.3 remains an open problem. While it is the simplest nontrivial case of Turán's problem, Erdös offered $500 for its solution.

Note that for each positive integer n, $ex(n, K_4^{(3)})$ is known to be at least the conjectured value in (1) corresponding to that equivalence class of n

modulo 3. We present the well known construction establishing this fact for the case when n is of the form $n = 3m$. Let V_1, V_2, V_3 be three partite sets of size m, and set $n = 3m$. Define

$$\begin{aligned}
\mathcal{G}_1 &= \{\{v_1, v_1', v_2\} : v_1 \in V_1, v_1' \in V_1, v_2 \in V_2\}, \\
\mathcal{G}_2 &= \{\{v_2, v_2', v_3\} : v_2 \in V_2, v_2' \in V_2, v_3 \in V_3\}, \\
\mathcal{G}_3 &= \{\{v_3, v_3', v_1\} : v_3 \in V_3, v_3' \in V_3, v_1 \in V_1\}, \\
\mathcal{G}_4 &= \{\{v_1, v_2, v_3\} : v_1 \in V_1, v_2 \in V_2, v_3 \in V_3\},
\end{aligned}$$

and define

$$\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4 \tag{3}$$

It follows by the construction in (3) that $\mathcal{G}$ is $K_4^{(3)}$-free and that

$$|\mathcal{G}| = \frac{m^2(5m-3)}{2} \tag{4}$$

Thus, $ex(n, K_4^{(3)}) \geq \frac{m^2(5m-3)}{2}$ when n is of the form $n = 3m$. Similar constructions establish that for all n, the conjectured value for $ex(n, K_4^{(3)})$ in (1) corresponding to the equivalence class of n modulo 3 is indeed a lower bound for $ex(n, K_4^{(3)})$.

Note that due to (4),

$$ex(K_4^{(3)}) \geq \frac{5}{9} = .555....$$

holds. However, we note that the construction above is not unique in achieving this lower bound. First alternative constructions were shown by Brown in [B], and subsequently in [K], Kostochka showed that there are 2^{m-2} such hypergraphs on $3m$ vertices.

The difficulty remaining in showing the equality in (2) is therefore in showing the upper bound $ex(K_4^{(3)}) \leq \frac{5}{9}$. The best known upper bound for $ex(K_4^{(3)})$ was given by Giraud (unpublished) and commented on in [De].

Theorem 1.4. (Giraud [1989])

$$ex(K_4^{(3)}) < \frac{\sqrt{21}-1}{6} = .597\ldots$$

Like Conjecture 1.3, Turán conjectured that other Turán numbers also had a "nice" answer. For example,

Conjecture 1.5. (Turán)

$$ex(K_5^{(3)}) = \frac{3}{4}$$

Additionally, the following open Turán type problem in [FF] should be easier than Conjecture 1.3.

Conjecture 1.6. (Frankl, Füredi) *Let $K_4^{(3)} - e$ be the triple system given by 3 triples on 4 vertices. Then*

$$ex(K_4^{(3)} - e) = \frac{2}{7}$$

Over the following two sections, we consider two general problems which relate to the problem of determining $ex(\mathcal{F})$ for various fixed l-uniform hypergraphs $\mathcal{F}$. In section 2, we present results and problems from [EFR] and recent results from [NR]. In section 3, we formulate some open problems and review related results from [CN] and [KRS].

2. A Problem Concerning Forbidding Families

We begin this section by considering an extremal problem for l-uniform hypergraphs concerning forbidden subhypergraphs. We start with the following definition.

Definition 2.1. For a fixed l-uniform hypergraph $\mathcal{F}$, let

$$Forb_n(\mathcal{F}) = \{\mathcal{G} \subseteq [n]^l : \mathcal{G} \text{ is } \mathcal{F}\text{-free } \}$$

and set

$$F_n(\mathcal{F}) = |Forb_n(\mathcal{F})|$$

Thus, $Forb_n(\mathcal{F})$ is the family of all l-uniform hypergraphs on vertex set $[n]$ which do not contain a copy of $\mathcal{F}$ as a subhypergraph, and $F_n(\mathcal{F})$ counts the number of such hypergraphs. Concerning the case when $l = 2$, the following result of [EFR] implies that for all non-bipartite graphs $\mathcal{F}$, the problem of determining $F_n(\mathcal{F})$ asymptotically is equivalent to determining $ex(\mathcal{F})$.

Theorem 2.2. (Erdös, Frankl, Rödl [1986]) *Suppose $\mathcal{F}$ is a fixed graph satisfying $\chi(\mathcal{F}) > 2$. Then,*

$$F_n(\mathcal{F}) = 2^{ex(n,\mathcal{F})(1+o(1))} \tag{5}$$

where $o(1) \to 0$ as $n \to \infty$.

We note that Theorem 2.2 is proved using Szemerédi's Regularity Lemma (c.f. [KS]). The proof is complicated, and we do not give it here.

It is remarked in [EFR] that (5) likely holds for bipartite graphs $\mathcal{F}$ as well, but this is not known even for $\mathcal{F} = C_4$, the cycle on 4 vertices. For this case, the best known upper bound $2^{cn^{3/2}}$ is due to Kleitman and Winston [KW].

The following extension of Theorem 2.2 for the case when $l = 3$ was recently obtained in [NR].

Theorem 2.3. (Nagle, Rödl [1999]) *Suppose $\mathcal{F}$ is an arbitrary fixed triple system. Then,*

$$F_n(\mathcal{F}) = 2^{ex(n,\mathcal{F})+o(n^3)}$$

where $\frac{o(n^3)}{n^3} \to 0$ *as* $n \to \infty$.

Similar to the assumption that graph $\mathcal{F}$ be non-bipartite in Theorem 2.2, we will assume for the following corollary that the fixed triple system $\mathcal{F}$ is not 3-partite (i.e. that for any partition of the vertices $V(\mathcal{F})$ of $\mathcal{F}$ into 3 classes V_1, V_2, V_3, there exists a triple $f \in \mathcal{F}$ and $i \in [3]$ such that $|f \cap V_i| \geq 2$). By an old result of Erdös in [E], a triple system $\mathcal{F}$ is 3-partite if and only if $ex(\mathcal{F}) = 0$. Thus, we have the following corollary to Theorem 2.3.

Corollary 2.4. *Suppose $\mathcal{F}$ is a fixed triple system which is not 3-partite. Then,*

$$F_n(\mathcal{F}) = 2^{ex(n,\mathcal{F})(1+o(1))} \tag{6}$$

where $o(1) \to 0$ *as* $n \to \infty$.

We note that Corollary 2.4 implies that, as in the case for graphs, the problem of determining $F_n(\mathcal{F})$ asymptotically for non 3-partite triple systems $\mathcal{F}$ is equivalent to determining $ex(\mathcal{F})$. We remark that Theorem 2.3 is proved using The Hypergraph Regularity Lemma of [FR] (see also [NR1]) in conjunction with the The Counting Lemma of [NR1]. The proof is complicated, and we encourage the Reader to see [NR].

We discuss a problem related to Theorem 2.4. We need, however, to first state the following definition. For two l-uniform hypergraphs $\mathcal{F}_1$, $\mathcal{F}_2$, a function $\psi : V(\mathcal{F}_1) \to V(\mathcal{F}_2)$ is called a *homomorphism* if for each $\{v_1, \ldots, v_l\} \in \mathcal{F}_1$, $\{\psi(v_1), \ldots, \psi(v_l)\} \in \mathcal{F}_2$. The following problem was raised in [EFR].

Problem 2.5. *Let $\mathcal{F}_1$ be an l-uniform hypergraph, and let $\mathcal{F}_2$ be any homomorphic image of $\mathcal{F}_1$. Is is true that for any $\epsilon > 0$, there exists $n_0 = n_0(\mathcal{F}_1, \mathcal{F}_2, \epsilon)$ so that for any $\mathcal{G} \in Forb_n(\mathcal{F}_1)$, $n \geq n_0$, there exists $\mathcal{G}' \subseteq \mathcal{G}$, $|\mathcal{G}'| < \epsilon n^3$, so that $\mathcal{G} \setminus \mathcal{G}' \in Forb_n(\mathcal{F}_2)$?*

They also provide the following theorem partially answering Problem 2.5.

Theorem 2.6. (Erdös, Frankl, Rödl [1986]) *Problem 2.5 is true for $l = 2$.*

Recently, in [NR], the following was proved.

Theorem 2.7. (Nagle, Rödl [1999]) *Problem 2.5 is true for l=3.*

We note that Theorem 2.6 was proved using Szemerédi's Regularity Lemma, and similarly, Theorem 2.7 was proved using The Hypergraph

Regularity Lemma of [FR], and The Counting Lemma of [NR1]. The proof of Theorem 2.7 is similar to the proof of Theorem 2.3.

Theorem 2.7 has the following corollary.

Corollary 2.8. *For any triple system $\mathcal{F}_1$, and any homomorphic image $\mathcal{F}_2$ of $\mathcal{F}_1$,*

$$ex(n, \mathcal{F}_2) \geq ex(n, \mathcal{F}_1) - o(n^3) \tag{7}$$

In [NR2], we pose the following problem concerning the possibility of equality holding in (7). In this problem, we use the term that a triple system $\mathcal{F}$ is *irreducible* to mean that every pair of vertices of $\mathcal{F}$ is contained in a triple of $\mathcal{F}$.

Problem 2.9. *For any non 3-partite triple system $\mathcal{F}_1$, does there exist an irreducible homomorphic image $\mathcal{F}_2$ such that*

$$ex(n, \mathcal{F}_2) = ex(n, \mathcal{F}_1)(1 + o(1))? \tag{8}$$

If equality were to hold in (8), it would provide a generalization for triple systems of the classical Erdös-Stone Theorem for graphs. Recall their theorem states the following (c.f. [ES]).

Theorem 2.10. (Erdös, Stone [1946]) *For any graph $\mathcal{F}$ with $\chi(\mathcal{F}) = r > 2$,*

$$ex(n, \mathcal{F}) = ex(n, K_r^{(2)})(1 + o(1))$$

Since complete graphs are precisely irreducible (in the sense that every pair of vertices is contained in an edge), $K_2^{(2)}$ is an irreducible homomorphic image of any graph $\mathcal{F}$ with $\chi(\mathcal{F}) = r$.

3. Pair Degree Turán Problems

In this section, all problems will concern triple systems. Before discussing these problems, we will need the following definition.

Definition 3.1. For a triple system $\mathcal{G}$ and two vertices $u, v \in V(\mathcal{G})$, let

$$codeg(\{u, v\}) = |\{g \in \mathcal{G} : \{u, v\} \subset g\}|$$

and set

$$codeg(\mathcal{G}) = min\left\{codeg(\{u, v\}) : \{u, v\} \in [V(\mathcal{G})]^2\right\}$$

The following problem was raised by S. Abassi in [A].

Problem 3.2. *For every $n \equiv 0$ (mod 4), if $\mathcal{G} \subseteq [n]^3$ satisfies $codeg(\mathcal{G}) \geq \frac{n}{2}$, does $\mathcal{G}$ admit a covering by vertex disjoint copies of $K_4^{(3)}$?*

In [CN], Problem 3.2 was answered negatively by the following theorem.

Theorem 3.3. **(Czygrinow, Nagle [1999])** *For every $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ so that for all $n \geq n_0$, there exists a triple system $\mathcal{G} \subset [n]^3$ satisfying the following properties:*

i. $codeg(\mathcal{G}) \geq \left(\frac{3}{5} - \epsilon\right) n$,

ii. *Every collection of vertex disjoint copies of $K_4^{(3)}$ contained in $\mathcal{G}$ leaves at least ϵn vertices uncovered.*

The proof of Theorem 3.3 is not difficult, but we do not give it here. We encourage the Reader to see the paper [CN].

Note that Theorem 3.3 can not be strengthened to assert that there exists triple systems $\mathcal{G}$ satisfying the condition in i. but which contain no copies of $K_4^{(3)}$. Indeed, by the condition in i., we are guarenteed that

$$\left(\frac{3}{5} - \epsilon\right) n \binom{n}{2} \leq \sum_{\{u,v\} \in [V(\mathcal{G})]^2} codeg(\{u,v\}) = 3|\mathcal{G}|, \tag{9}$$

hence it follows from Theorem 1.4 that for sufficiently large n,

$$|\mathcal{G}| \geq \left(\frac{3}{5} - \epsilon\right) \frac{n}{3} \binom{n}{2} > ex(n, K_4^{(3)}) \tag{10}$$

So in particular, the condition in i. guarentees the existence of at least one copy of $K_4^{(3)}$.

Problem 3.2, while not true, suggests the following problem: *Do the conditions of Problem 3.2 guarentee that $\mathcal{G}$ contains even 1 copy of $K_4^{(3)}$?* This problem is still open, and appears to be difficult. We discuss this problem more generally below. Before doing so, we require the following definition.

Definition 3.4. For a fixed triple system $\mathcal{F}$, define

$$c(\mathcal{F}) = inf\,\{c : \text{ For all triple systems } \mathcal{G},\ codeg(\mathcal{G}) \geq c|V(\mathcal{G})| \Rightarrow \mathcal{F} \subseteq \mathcal{G}\}$$

Similar to the calculations in (9) and (10), one can conclude the following easy upper bound for $c(\mathcal{F})$ for all $\mathcal{F}$

$$c(\mathcal{F}) \leq ex(\mathcal{F})$$

Analogously to Conjectures 1.3 and 1.6, one can make the following conjectures.

Conjecture 3.5. *Is it true that*

$$c(K_4^{(3)}) = \frac{1}{2}?$$

Conjecture 3.6. *Let $K_4^{(3)} - e$ be that triple system given by 3 triples on 4 points. Is it true that*

$$c(K_4^{(3)} - e) = \frac{1}{4}?$$

Note that similarly to Conjectures 1.3 and 1.6, the lower bounds are known. They follow from constructions considered by Erdös, Hajnal, Sós and Simonovits and also by Rödl (c.f. [E1], [R]).

In [KRS], the authors were able to show a statement related to the upper bound of $c(K_4^{(3)} - e)$ which solved a Turán-Ramsey problem. If a triple system $\mathcal{G}$ satisfies $codeg(\mathcal{G}) \geq \frac{1}{4}$ *and* one assumes that the triples of $\mathcal{G}$ assume a very regular distribution (i.e. as though they had been generated "randomly") with positive density, then $\mathcal{G}$ must contain a copy of $K_4^{(3)} - e$. For more information on their statement, see [KRS]. On the other hand, concerning the same problem for the triple system $K_4^{(3)}$, we note here that even under the similar conditions that $\mathcal{G}$ satisfies $codeg(\mathcal{G}) \geq \frac{1}{2}$ and the triples of $\mathcal{G}$ assume a very regular distribution with positive density (as in [KRS]), it is still not known whether $\mathcal{G}$ must contain a copy of $K_4^{(3)}$.

4. Acknowledgements

The author wishes to thank A. Czygrinow and V. Rödl for stimulating discussion on Turán problems for hypergraphs.

5. References

[A] S. Abassi, Personal communication with V. Rödl

[B] W.G. Brown, *On an open problem of Paul Turán concerning 3-graphs*, in Studies in Pure Mathematics, 91-93. Basel-Boston: Birkhäuser, 1983

[CN] A. Czygrinow, B. Nagle, *A Turán type problem concerning pair degree restrictions*, manuscript

[De] D. de Caen, *The Current Status of Turán's problem on hypergraphs*, in Extremal Problems for Finite Sets *(Visegrád, 1991)*. Bolyai Soc. Math. Stud., Volume 3, 187-189. Budapest: János Bolyai Math. Soc., 1991.

[E] P. Erdös, *On extremal problems of graphs and generalized graphs*, Israel J. of Math. 2, 189-190 (1965)

[E1] P. Erdös, *Problems and results on graphs and hypergraphs: similarities and differences*, in Mathematics of Ramsey Theory, J. Nešetril and V. Rödl (eds.), 12-25. Algorithms and Combinatorics 5, Springer Verlag, 1990

[EFR] P. Erdös, P. Frankl, V. Rödl, *The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs have no exponent*, Graphs and Combinatorics 2, 113-121 (1986)

[ES] P. Erdös, A. H. Stone, *On the structure of linear graphs*, J Bull. Amer. Math. Soc., 52, 1087-1091 (1946)

[FF] P. Frankl, Z. Füredi, *An exact result for 3-graphs*, Discrete Math. 50, 323-328

[FR] P. Frankl, V. Rödl, *Extremal problems on set systems*, submitted.

[KRS] Y. Kohayakawa, V. Rödl, E. Szemerédi, *Turán-Ramsey Problem for 3-uniform hypergraphs*, manuscript

[KS] J. Komlós, M. Simonovits, *Szemerédi's Regularity Lemma and its application in graph theory*, In: Combinatorics, Paul Erdös is eighty, Volume 2, János Bolyai Mathematical Society, Budapest, Hungary, 295-352 (1996)

[KW] D. J. Kleitman, K. J. Winston, *The asymptotic number of lattices*, In: Combinatorial Mathematics, Optimal Designs and their Applications, edited by J. Shrivastava, Ann. Discrete Math. 6, 243-249, (1980)

[NR] B. Nagle, V. Rödl, *The asymptotic number of triple systems not containing a fixed one*, submitted

[NR1] B. Nagle, V. Rödl, *Regularity properties for triple systems*, manuscript

[R] V. Rödl, *On universality of graphs with uniformly distributed edges*, Discrete Math. 59, 125-134

[T] P. Turán, *An extremal problem in graph theory (in Hungarian)*, Mat. Fiz. Lapok 48, 436-452 (1941)