An Algorithmic Hypergraph Regularity Lemma

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Abstract

Szemerédi's Regularity Lemma [22, 23] is a powerful tool in graph theory. It asserts that all large graphs G admit a bounded partition of $E(G)$, most classes of which are bipartite subgraphs with uniformly distributed edges. The original proof of this result was non-constructive. A constructive proof was given by Alon, Duke, Lefmann, Rödl and Yuster [1], which allows one to efficiently construct a regular partition for any large graph.

Szemerédi's Regularity Lemma was extended to hypergraphs by various authors. Frankl and Rödl [3] gave one such extension to 3-uniform hypergraphs, and Rödl and Skokan [19] extended this result to k-uniform hypergraphs. W.T. Gowers [4, 5] gave another such extension. Similarly to the graph case, all of these proofs are non-constructive. We present an efficient algorithmic version of the Hypergraph Regularity Lemma for k-uniform hypergraphs.

1 Introduction

Szemerédi's Regularity Lemma [22, 23] is an important tool in combinatorics, with applications in combinatorial number theory, extremal graph theory, and theoretical computer science (see [9, 10] for surveys of applications). The Regularity Lemma hinges on the notion of ε-regularity: a bipartite graph $H = (X \cup Y, E)$ is *ε-regular* if for every $X' \subseteq X$, $|X'| > \varepsilon|X|$, and for every $Y' \subseteq Y$, $|Y'| > \varepsilon|Y|$, we have $|d_H(X', Y') - d_H(X, Y)| < \varepsilon$, where $d_H(X', Y') = |H[X', Y']|/(|X'||Y'|)$ is the *density* of the bipartite subgraph $H[X', Y']$ induced on $X' \cup Y'$. Szemerédi's Regularity Lemma is then stated as follows.

Theorem 1.1 *For all $\varepsilon > 0$ and $t_0 \in \mathbb{N}$, there exist integers $T_0 = T_0(\varepsilon, t_0)$ and $N_0 = N_0(\varepsilon, t_0)$ so that every graph G on $N > n_0$ vertices admits a partition $V(G) = V_1 \cup \cdots \cup V_t$, $t_0 \leq t \leq T_0$, satisfying that*

1. *$V(G) = V_1 \cup \cdots \cup V_t$ is equitable, meaning that $|V_1| \leq \cdots \leq |V_t| \leq |V_1| + 1$;*

2. *$V(G) = V_1 \cup \cdots \cup V_t$ is ε-regular, meaning that all but $\varepsilon\binom{t}{2}$ pairs V_i, V_j, $1 \leq i < j \leq t$, are ε-regular.*

The original proof of Theorem 1.1 was non-constructive. A constructive proof of Theorem 1.1 was given by Alon, Duke, Lefmann, Rödl and Yuster [1]. Their result gives that the ε-regular partition $V(G) = V_1 \cup \cdots \cup V_t$ in Theorem 1.1 can be constructed in time $O(M(N))$, where $M(N)$ is the time needed to multiply two $N \times N$ matrices with $0, 1$-entries over the integers, and for which it was recently shown that $M(N) = O(N^{2.3727})$ (see [24]). In [8], an algorithmic version with optimal running time $O(N^2)$ was established.

Szemerédi's Regularity Lemma has been extended to k-uniform hypergraphs, for $k \geq 2$, by various authors. Frankl and Rödl [3] gave one such extension to 3-uniform hypergraphs, using a concept of regularity they called *(δ, r)-regularity* (see upcoming Definition 2.9). This regularity lemma was extended to k-uniform hypergraphs, for arbitrary $k \geq 3$, by Rödl and Skokan [19]. Gowers [4, 5] also established a regularity lemma for

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k-uniform hypergraphs, using a concept of regularity known as *deviation* (see upcoming Definition 2.6). While the concepts of (δ, r)-regularity and deviation are different, the corresponding Regularity Lemmas have a similar conclusion and allow similar applications. Roughly speaking, both lemmas guarantee that every (large) k-uniform hypergraph $\mathcal{H}^{(k)}$ admits a bounded partition of $E(\mathcal{H}^{(k)})$, where most classes of the partition consist of 'regularly distributed' edges. Moreover, both Regularity Lemmas admit a corresponding *Counting Lemma* [4, 5, 13] (not stated in this paper), which estimates the number of fixed subhypergraphs of a given isomorphism type within the 'regular partition' provided by the regularity lemma. The combined application of the Regularity and Counting Lemmas is known as the *Regularity Method* for hypergraphs (see [14, 17, 18, 21]).

Similarly to the graph case, the original proofs of the hypergraph regularity lemma are non-constructive. Here we present a constructive proof of Gowers' Hypergraph Regularity Lemma (see upcoming Theorem 3.5). Thus, combining the work here together with Gowers' Counting Lemma provides an *Algorithmic Regularity Method* for hypergraphs. At the end of the Introduction, we discuss an application of this method. (For an algorithmic regularity method for 3-uniform hypergraphs, see [6, 7, 12].)

To prove the algorithmic regularity lemma for hypergraphs, we will proceed along the usual lines. As in the proof of Szemerédi [22, 23] for graphs, we will consider sequences of partitions $\mathscr{P}_i$, $i \geq 1$, of a hypergraph $\mathcal{H}^{(k)}$. (In fact, each $\mathscr{P}_i$, $i \geq 1$, is a family $\mathscr{P}_i = \{\mathscr{P}_i^{(1)}, \ldots, \mathscr{P}_i^{(k-1)}\}$ of partitions of vertices, pairs, ..., $(k-1)$-tuples of $V(\mathcal{H}^{(k)})$.) For each $\mathscr{P}_i$, $i \geq 1$, we consider a so-called *index* of $\mathscr{P}_i$, denoted $\mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}_i)$, which measures the mean-square density of $\mathcal{H}^{(k)}$ on $\mathscr{P}_i$. When the partition $\mathscr{P}_i$ of $\mathcal{H}^{(k)}$ is irregular, we refine $\mathscr{P}_i$, in the usual way, to produce $\mathscr{P}_{i+1}$. It is well-known that $\mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}_{i+1})$ will be non-negligibly larger than $\mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}_i)$, so that this refining process must terminate after constantly many iterations. Now, as in the proof of Alon et al. [1] for graphs, to make the the refinement $\mathscr{P}_{i+1}$ of $\mathscr{P}_i$ constructive, one must be able to *construct* 'witnesses' of the irregularity of $\mathscr{P}_i$. The novel element of our proof does precisely this, which we call the *Witness-Construction Theorem* (upcoming Theorem 2.12). Due to its significant technicality, we make no effort to outline the proof of Theorem 2.12 in this abstract. We refer the Reader to the full version [15] for details.

The remainder of this paper is organized as follows. In Section 2, we present the concept of deviation, and we present Theorem 2.12. In Section 3, we present Theorem 3.5. In Section 4, we outline how Theorem 3.5 may be inferred from Theorem 2.12.

We conclude our current discussion with the following application of the algorithmic regularity method for hypergraphs. It is well-known that for a given graph $G = (V = [n], E)$, one may count its number of triangles $|\mathcal{K}_3(G)|$ in time much faster than $O(n^3)$. Indeed, let $A = [a_{ij}]_{1 \leq i,j \leq n}$ be the adjacency matrix of G, and set $B = A^2 = [b_{ij}]_{1 \leq i,j \leq n}$. Then, it is easy to see that

$$|\mathcal{K}_3(G)| = \frac{1}{3} \sum_{\{i,j\} \in E} b_{ij},$$

and so, by definition, one may compute $|\mathcal{K}_3(G)|$ in time $O(M(n)) = O(n^{2.3727})$. For fixed but arbitrary $f \geq 3$, Nešetřil and Poljak [16] extended the approach above to count $|\mathcal{K}_f(G)|$, the number of f-cliques in G, in time $O(n^{\omega \lfloor f/3 \rfloor + (f \bmod 3)})$, where ω denotes the exponent in $M(n)$, i.e., currently $\omega = 2.3727$. Using the Szemerédi Regularity Lemma, Kohayakawa, Rödl and Thoma [8] (cf. [2]) showed that $|\mathcal{K}_f(G)|$ can be approximated, within an additive error of $o(n^f)$, in time $O(n^2)$.

For integers $f > k \geq 3$, and for an n-vertex k-graph $\mathcal{H}^{(k)}$, Yuster [25] asked if there is an algorithm which counts $|\mathcal{K}_f(\mathcal{H}^{(k)})|$, the number of f-cliques in $\mathcal{H}^{(k)}$, in time $o(n^f)$. The current best running time for this problem is $O(n^f/\log n)$, which is due to Nagle [11]. Using the methods discussed in this paper, one may extend the result of [8] to the hypergraph setting, which is our promised application.

Theorem 1.2 *Let integers $f \geq k \geq 2$ and $\gamma > 0$ be given. There exists $N_0 = N_0(f, k, \gamma)$ so that for any k-graph $\mathcal{H}^{(k)}$ on $n > N_0$ vertices, the quantity $|\mathcal{K}_f(\mathcal{H}^{(k)})|$ may be approximated, in time $O(n^{3k})$, within an additive error of γn^f.*

Theorem 1.2 is actually a straightforward consequence of the algorithmic regularity method for hypergraphs. For the proof, we refer the Reader to [6], where this proof is sketched for $k = 3$. We mention that Theorem 1.2 may be easily generalized to estimate, for any fixed subhypergraph $\mathcal{F}^{(k)}$, the number of induced copies of $\mathcal{F}^{(k)}$ in $\mathcal{H}^{(k)}$, also in time $O(n^{3k})$.

It remains an interesting question if, for hypergraphs, an extension of the Nešetřil-Poljak theorem [16] is possible, even for $k=3$: does there exist $\varepsilon > 0$, $f_0 \in \mathbb{N}$, and an algorithm with running time $O(n^{(1-\varepsilon)f})$ that determines the number of cliques $K_f^{(3)}$ (for $f \geq f_0$) in a given n-vertex 3-graph $\mathcal{H}^{(3)}$?

2 Deviation and the Witness-Construction Theorem

In this section, we define the concept of *deviation* (**DEV**) (cf. Definition 2.6). We also consider the concept of *r-discrepancy* (r-**DISC**) (cf. Definition 2.9), and present a so-called Witness-Construction theorem (cf. Theorem 2.12). For these purposes, we need some supporting concepts.

2.1 Background concepts: cylinders, complexes and density

We begin with some basic concepts. For a set X and an integer $j \leq |X|$, let $\binom{X}{j}$ denote the set of all (unordered) j-tuples from X. When $X = [\ell] = \{1, \ldots, \ell\}$, we sometimes write $[\ell]^j = \binom{[\ell]}{j}$. Given pairwise disjoint sets $V_1, \ldots, V_\ell$, denote by $K^{(j)}(V_1, \ldots, V_\ell)$ the *complete* ℓ-partite, j-uniform hypergraph with ℓ-partition $V_1 \cup \cdots \cup V_\ell$, which consists of all j-tuples from $V_1 \cup \cdots \cup V_\ell$ meeting each V_a, $1 \leq a \leq j$, at most once. We now define the concept of a 'cylinder'.

Definition 2.1 (cylinder) For integers $\ell \geq j \geq 1$, an (ℓ, j)-*cylinder* $\mathcal{H}^{(j)}$ with vertex ℓ-partition $V(\mathcal{H}^{(j)}) = V_1 \cup \cdots \cup V_\ell$ is any subset of $K^{(j)}(V_1, \ldots, V_\ell)$. When $|V_1| = \cdots = |V_\ell| = m$, we say $\mathcal{H}^{(j)}$ is an (m, ℓ, j)-*cylinder.*

In the context of Definition 2.1, fix $j \leq i \leq \ell$ and $\Lambda_i \in [\ell]^i$. We denote by $\mathcal{H}^{(j)}[\Lambda_i] = \mathcal{H}^{(j)}\big[\bigcup_{\lambda\in\Lambda_i} V_\lambda\big]$ the sub-hypergraph of the (ℓ, j)-cylinder $\mathcal{H}^{(j)}$ induced on $\bigcup_{\lambda\in\Lambda_i} V_\lambda$. In this setting, $\mathcal{H}^{(j)}[\Lambda_i]$ is an (i, j)-cylinder.

We now prepare to define the concept of a complex. For an integer $i \geq j$, let $\mathcal{K}_i(\mathcal{H}^{(j)})$ denote the family of all i-element subsets of $V(\mathcal{H}^{(j)})$ which span complete sub-hypergraphs in $\mathcal{H}^{(j)}$. Given an $(\ell, j-1)$-cylinder $\mathcal{H}^{(j-1)}$ and an (ℓ, j)-cylinder $\mathcal{H}^{(j)}$, we say $\mathcal{H}^{(j-1)}$ *underlies* $\mathcal{H}^{(j)}$ if $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$.

Definition 2.2 (complex) For integers $1 \leq k \leq \ell$, an (ℓ, k)-*complex* $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ is a collection of (ℓ, j)-cylinders, $1 \leq j \leq k$, so that

1. $\mathcal{H}^{(1)} = V_1 \cup \cdots \cup V_\ell$ is an $(\ell, 1)$-cylinder, i.e., is an ℓ-partition;
2. for each $2 \leq j \leq k$, we have that $\mathcal{H}^{(j-1)}$ underlies $\mathcal{H}^{(j)}$, i.e., $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$.

In some cases, we use the notation $\boldsymbol{\mathcal{H}}^{(k)}$ to denote an (ℓ, k)-complex $\{\mathcal{H}^{(j)}\}_{j=1}^k$.

We now define concept of density.

Definition 2.3 (density) For integers $2 \leq j \leq \ell$, let $\mathcal{H}^{(j)}$ be an (ℓ, j)-cylinder and let $\mathcal{H}^{(j-1)}$ be an $(\ell, j-1)$-cylinder. If $\mathcal{K}_j(\mathcal{H}^{(j-1)}) \neq \varnothing$, we define the *density of* $\mathcal{H}^{(j)}$ *w.r.t.* $\mathcal{H}^{(j-1)}$ as

$$d(\mathcal{H}^{(j)}|\mathcal{H}^{(j-1)}) = \frac{\big|\mathcal{H}^{(j)} \cap \mathcal{K}_j(\mathcal{H}^{(j-1)})\big|}{\big|\mathcal{K}_j(\mathcal{H}^{(j-1)})\big|}.$$

If $\mathcal{K}_j(\mathcal{H}^{(j-1)}) = \varnothing$, we define $d(\mathcal{H}^{(j)}|\mathcal{H}^{(j-1)}) = 0$.

2.2 Deviation

In this subsection, we define the concept of deviation (**DEV**). To that end, we need some supporting concepts.

Definition 2.4 ((ℓ, j)-octohedron) Let integers $1 \leq j \leq \ell$ be given. The (ℓ, j)-*octohedron* $\mathcal{O}^{(j)} = \mathcal{O}_\ell^{(j)}$ is the complete ℓ-partite j-uniform hypergraph $K^{(j)}(U_1, \ldots, U_\ell)$, where $|U_1| = \cdots = |U_\ell| = 2$, i.e., it is the complete $(2, \ell, j)$-cylinder.

For an (ℓ, j)-cylinder $\mathcal{H}^{(j)}$, we are interested in 'labeled partite-embedded' copies of $\mathcal{O}^{(j)}$ in $\mathcal{H}^{(j)}$.

Definition 2.5 (labeled partite-embedding) Let $\mathcal{H}^{(j)}$ be an (ℓ, j)-cylinder, with ℓ-partition $V(\mathcal{H}^{(j)}) = V_1 \cup \cdots \cup V_\ell$, and let $\mathcal{O}^{(j)} = K^{(j)}(U_1, \ldots, U_\ell)$ be the (ℓ, j)-octohedron. A *labeled, partite-embedding* of $\mathcal{O}^{(j)}$ in $\mathcal{H}^{(j)}$ is an edge-preserving injection $\psi : U_1 \cup \cdots \cup U_\ell \to V_1 \cup \cdots \cup V_\ell$ so that $\psi(U_i) \subseteq V_i$ for each $1 \leq i \leq \ell$. We write $\mathrm{EMB}_{\mathrm{part}}(\mathcal{O}^{(j)}, \mathcal{H}^{(j)})$ to denote the family of all labeled partite-embeddings ψ of $\mathcal{O}^{(j)}$ in $\mathcal{H}^{(j)}$.

We now define the concept of deviation.

Definition 2.6 (deviation (DEV)) Let $\mathcal{H}^{(j)}$ be a (j,j)-cylinder with underlying $(j,j-1)$-cylinder $\mathcal{H}^{(j-1)}$. Let $\mathcal{H}^{(j)}$ and $\mathcal{H}^{(j-1)}$ have common vertex j-partition $V(\mathcal{H}^{(j)}) = V(\mathcal{H}^{(j-1)}) = V_1 \cup \cdots \cup V_j$, and let $d = d(\mathcal{H}^{(j)}|\mathcal{H}^{(j-1)})$. For $\delta > 0$, we say that $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ has (d,δ)-*deviation*, written $\mathbf{DEV}(d,\delta)$, if

$$\sum_{v_1,v_1'\in V_1} \cdots \sum_{v_j,v_j'\in V_j} \prod \Big\{ \omega(J) : J \in K^{(j)}\left(\{v_1,v_1'\},\dots,\{v_j,v_j'\}\right) \Big\} \le \delta \big|\mathrm{EMB}_{\mathrm{part}}(\mathcal{O}^{(j-1)}, \mathcal{H}^{(j-1)})\big|,$$

where for every $v_1, v_1' \in V_1, \dots, v_j, v_j' \in V_j$, and for each $J \in K^{(j)}\left(\{v_1,v_1'\},\dots,\{v_j,v_j'\}\right)$,

$$\omega(J) = \begin{cases} 1-d & \text{if } J \in \mathcal{H}^{(j)}, \\ -d & \text{if } J \in \mathcal{K}_j(\mathcal{H}^{(j-1)}) \setminus \mathcal{H}^{(j)}, \\ 0 & \text{if } J \notin \mathcal{K}_j(\mathcal{H}^{(j-1)}). \end{cases}$$

It is easy to extend Definition 2.6 from (j,j)-cylinders to (ℓ,k)-complexes.

Definition 2.7 Let $\boldsymbol{\delta} = (\delta_2,\dots,\delta_k)$ and $\boldsymbol{d} = (d_{\Lambda_j} : \Lambda_j \in [\ell]^j, 2 \le j \le k)$ be sequences of positive reals, and let (ℓ,k)-complex $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ be given. We say the complex $\boldsymbol{\mathcal{H}}$ has $\mathbf{DEV}(\boldsymbol{d},\boldsymbol{\delta})$ if, for each $2 \le j \le h$ and $\Lambda_j \in [\ell]^j$, $(\mathcal{H}^{(j)}[\Lambda_j], \mathcal{H}^{(j-1)}[\Lambda_j])$ has $\mathbf{DEV}(d_{\Lambda_j}, \delta_j)$.

2.3 r-discrepancy, and the Witness-Construction Theorem

In this subsection, we define the concept of r-discrepancy (r-**DISC**), and present the Witness Construction Theorem (cf. Theorem 2.12). We begin with the following extension of the concept of density (cf. Definition 2.3).

Definition 2.8 (r-density) Let $\mathcal{H}^{(j)}$ and $\mathcal{H}^{(j-1)}$ be given as in Definition 2.3, and let integer $r \ge 1$ be given. Let $\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$ satisfy $\bigcup_{i\in[r]} \mathcal{K}_j(\mathcal{Q}_i^{(j-1)}) \neq \varnothing$. We define the r-*density of* $\mathcal{H}^{(j)}$ *w.r.t.* $\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)}$ as

$$d(\mathcal{H}^{(j)}|\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)}) = \frac{\big|\mathcal{H}^{(j)} \cap \bigcup_{i\in[r]} \mathcal{K}_j(\mathcal{Q}_i^{(j-1)})\big|}{\big|\bigcup_{i\in[r]} \mathcal{K}_j(\mathcal{Q}_i^{(j-1)})\big|}.$$

We now define the concept of r-discrepancy.

Definition 2.9 (r-discrepancy (r-DISC)) Let $\mathcal{H}^{(j)}$ and $\mathcal{H}^{(j-1)}$ be given as in Definition 2.3, where $d = d(\mathcal{H}^{(j)}|\mathcal{H}^{(j-1)})$. For $\delta > 0$ and an integer $r \ge 1$, we say that $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ has (d,δ,r)-*discrepancy*, written $\mathbf{DISC}(d,\delta,r)$, if for any collection $\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$,

$$\Big| \bigcup_{i\in[r]} \mathcal{K}_j(\mathcal{Q}_i^{(j-1)}) \Big| > \delta |\mathcal{K}_j(\mathcal{H}^{(j-1)})| \implies \Big| d(\mathcal{H}^{(j)}|\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)}) - d \Big| < \delta. \quad (1)$$

For brevity, we sometimes refer to (d,δ,r)-discrepancy as r-*discrepancy*, and sometimes write $\mathbf{DISC}(d,\delta,r)$ as r-**DISC**.

We will also need the following concept, related to Definition 2.9.

Definition 2.10 (r-witness) Let $\mathcal{H}^{(j)}$ and $\mathcal{H}^{(j-1)}$ be given as in Definition 2.9, where $d = d(\mathcal{H}^{(j)}|\mathcal{H}^{(j-1)})$. Suppose that $(\mathcal{H}^{(j)}, \mathcal{H}^{(j-1)})$ does not have $\mathbf{DISC}(d,\delta,r)$, for some $\delta > 0$ and integer $r \ge 1$. We call any collection $\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$ for which

$$\Big| \bigcup_{i\in[r]} \mathcal{K}_j(\mathcal{Q}_i^{(j-1)}) \Big| > \delta |\mathcal{K}_j(\mathcal{H}^{(j-1)})| \quad \text{but} \quad \Big| d(\mathcal{H}^{(j)}|\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)}) - d \Big| \ge \delta.$$

an r-*witness* of $\neg\,\mathbf{DISC}(d,\delta,r)$.

We finally present the Witness-Construction Theorem, which concerns a (k,k)-complex $\boldsymbol{\mathcal{H}}$ satisfying the following setup.

Setup 2.11 Let $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}^{(k)} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ be a (k,k)-complex, where $\mathcal{H}^{(1)} = V_1 \cup \cdots \cup V_k$ has $m \le |V_i| \le m+1$ for all $i \in [k]$. Let

$$\boldsymbol{d}_k = \left(d_{\Lambda_j} : \Lambda_j \in [k]^j, 2 \le j \le k\right)$$

satisfy that, for each $2 \le j \le k$ and for each $\Lambda_j \in [k]^j$,

$$d_{\Lambda_j} = d(\mathcal{H}^{(j)}[\Lambda_j]|\mathcal{H}^{(j-1)}[\Lambda_j]).$$

Note, in particular, that $d_{[k]} = d(\mathcal{H}^{(k)}|\mathcal{H}^{(k-1)})$. We call $\boldsymbol{d}_k$ the *density sequence* for $\boldsymbol{\mathcal{H}}^{(k)}$. Write $\boldsymbol{\mathcal{H}}^{(k-1)} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1}$ and

$$\boldsymbol{d}_{k-1} = \left(d_{\Lambda_j} : \Lambda_j \in [k]^j, 2 \le j \le k-1\right),$$

so that $\boldsymbol{d}_{k-1}$ is the density sequence for $\boldsymbol{\mathcal{H}}^{(k-1)}$.

The Witness-Construction Theorem is now given as follows.

Theorem 2.12 (Witness-Construction) *Let integer $k \geq 2$ be fixed. For all $d_k, \delta_k > 0$, there exists $\delta'_k > 0$ so that for all $d_{k-1} > 0$, there exists $\delta_{k-1} > 0$ so that, …, for all $d_2 > 0$, there exist $\delta_2 > 0$, positive integer r_0, and positive integer m_0 so that the following holds.*

Set $\boldsymbol{\delta}_{k-1} = (\delta_2, \dots, \delta_{k-1})$. Let $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}^{(k)}$ be a (k,k)-complex with density sequence $\boldsymbol{d}_k$, as given as in Setup 2.11, where $m \geq m_0$. Suppose $\boldsymbol{d}_k$ satisfies that, for each $2 \leq j \leq k$ and for each $\Lambda_j \in [k]^j$, $d_{\Lambda_j} \geq d_j$. Assume that

1. *$\boldsymbol{\mathcal{H}}^{(k-1)}$ has $\mathbf{DEV}(\boldsymbol{d}_{k-1}, \boldsymbol{\delta}_{k-1})$, but that*
2. *$(\mathcal{H}^{(k)}, \mathcal{H}^{(k-1)})$ does not have $\mathbf{DEV}(d_{[k]}, \delta_k)$.*

Then, there exists an algorithm which constructs, in time $O(m^{3k})$, an r-witness $\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)} \subseteq \mathcal{H}^{(k-1)}$ of $\neg\mathbf{DISC}(d_{[k]}, \delta'_k, r)$, for some $r \leq r_0$.

3 Algorithmic Hypergraph Regularity Lemma

In this section, we state an Algorithmic Hypergraph Regularity Lemma (see Theorem 3.5, below) for the property of deviation. To state this lemma, we still need some more concepts.

3.1 Families of partitions

Theorem 3.5 provides a well-structured family of partitions $\mathscr{P} = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ of vertices, pairs, …, and $(k-1)$-tuples of a given vertex set. We will define the properties of $\mathscr{P}$ in upcoming Definitions 3.1 and 3.2, but we first need to establish some notation and concepts.

We first discuss the structure of these partitions inductively, following the approach of [13]. Let k be a fixed integer and V be a set of vertices. Let $\mathscr{P}^{(1)} = \{V_1, \dots, V_{|\mathscr{P}^{(1)}|}\}$ be a partition of V. For every $1 \leq j \leq |\mathscr{P}^{(1)}|$, let $\mathrm{Cross}_j(\mathscr{P}^{(1)}) = K^{(j)}(V_1, \dots, V_{|\mathscr{P}^{(1)}|})$ be the family of all crossing j-tuples J, i.e., the set of j-tuples which satisfy $|J \cap V_i| \leq 1$ for every $1 \leq i \leq |\mathscr{P}^{(1)}|$.

Suppose that partitions $\mathscr{P}^{(i)}$ of $\mathrm{Cross}_i(\mathscr{P}^{(1)})$ have been defined for all $1 \leq i \leq j-1$. Then for every $I \in \mathrm{Cross}_{j-1}(\mathscr{P}^{(1)})$, there exists a unique class $\mathcal{P}^{(j-1)} = \mathcal{P}^{(j-1)}(I) \in \mathscr{P}^{(j-1)}$ so that $I \in \mathcal{P}^{(j-1)}$. For every $J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})$, we define the *polyad of* J by $\hat{\mathcal{P}}^{(j-1)}(J) = \bigcup \{\mathcal{P}^{(j-1)}(I) \colon I \in [J]^{j-1}\}$. Define the family of all polyads $\hat{\mathscr{P}}^{(j-1)} = \{\hat{\mathcal{P}}^{(j-1)}(J) \colon J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})\}$, which we view as a set (as opposed to a multiset, since $\hat{\mathcal{P}}^{(j-1)}(J) = \hat{\mathcal{P}}^{(j-1)}(J')$ may hold for $J \neq J'$). To simplify notation, we often write the elements of $\hat{\mathscr{P}}^{(j-1)}$ as $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$ (dropping the argument J).

Observe that $\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ is a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$. The structural requirement on the partition $\mathscr{P}^{(j)}$ of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ is

$$\mathscr{P}^{(j)} \prec \{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}, \quad (2)$$

where '$\prec$' denotes the refinement relation of set partitions. Note that (2) inductively implies that

$$\boldsymbol{\mathcal{P}}(J) = \{\hat{\mathcal{P}}^{(i)}(J)\}_{i=1}^{j-1}, \quad \text{where } \hat{\mathcal{P}}^{(i)}(J) = \bigcup \{\mathcal{P}^{(i)}(I) \colon I \in [J]^i\}, \quad (3)$$

is a $(j, j-1)$-complex (since each $\hat{\mathcal{P}}^{(i)}(J)$ is a (j,i)-cylinder). We may now give Definitions 3.1 and 3.2.

Definition 3.1 ($\boldsymbol{a}$-family of partitions) Let V be a set of vertices, and let $k \geq 2$ be a fixed integer. Let $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ be a sequence of positive integers. We say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}) = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ is an *$\boldsymbol{a}$-family of partitions on V*, if it satisfies the following:

(*a*) $\mathscr{P}^{(1)}$ is a partition of V into a_1 classes,

(*b*) $\mathscr{P}^{(j)}$ is a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ refining $\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ where, for every $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$, $|\{\mathcal{P}^{(j)} \in \mathscr{P}^{(j)} \colon \mathcal{P}^{(j)} \subseteq \mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\}| = a_j$.

Moreover, we say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ is *t-bounded*, if $\max\{a_1, \dots, a_{k-1}\} \leq t$.

3.2 Properties of families of partitions

In this subsection, we describe some properties we would like an $\boldsymbol{a}$-family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ to have.

Definition 3.2 ($(\eta, \boldsymbol{\delta}, \geq\!\boldsymbol{D}, \boldsymbol{a})$-family) Let V be a set vertices, let $\eta > 0$ be fixed, and let $k \geq 2$

be a fixed integer. Let $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_{k-1})$ and $\boldsymbol{D} = (D_2, \ldots, D_{k-1})$ be sequences of positives, and let $\boldsymbol{a} = (a_1, \ldots, a_{k-1})$ be a sequence of positive integers.

We say an $\boldsymbol{a}$-family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ on V is an $(\eta, \boldsymbol{\delta}, \geq\boldsymbol{D}, \boldsymbol{a})$-*family* if it satisfies the following conditions:

(a) $\mathscr{P}^{(1)} = \{V_i \colon i \in [a_1]\}$ is an equitable vertex partition, i.e., $\lfloor |V|/a_1 \rfloor \leq |V_i| \leq \lceil |V|/a_1 \rceil$ for $i \in [a_1]$;

(b) $\big| [V]^k \setminus \mathrm{Cross}_k(\mathscr{P}^{(1)}) \big| \leq \eta |V|^k$;

(c) all but $\eta |V|^k$ many k-tuples $K \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$ satisfy that for each $2 \leq j \leq k-1$, and for each $J \in \binom{K}{j}$, the pair $(\mathcal{P}^{(j)}(J), \hat{\mathcal{P}}^{(j-1)}(J))$ has $\mathbf{DEV}(d_J, \delta_j)$, where $d_J = d(\mathcal{P}^{(j)}(J) | \hat{\mathcal{P}}^{(j-1)}(J)) \geq D_j$.

Note that in an $(\eta, \boldsymbol{\delta}, \geq\boldsymbol{D}, \boldsymbol{a})$-family of partitions $\mathscr{P}$ on V, properties (b) and (c) above imply that all but $2\eta|V|^k$ many k-tuples $K \in [V]^k$ belong to $\mathrm{Cross}_k(\mathscr{P}^{(1)})$ and satisfy that, for each $2 \leq j \leq k-1$, and for each $J \in \binom{K}{j}$, the pair $(\mathcal{P}^{(j)}(J), \hat{\mathcal{P}}^{(j-1)}(J))$ has $\mathbf{DEV}(d_J, \delta_j)$, where $d_J = d(\mathcal{P}^{(j)}(J) | \hat{\mathcal{P}}^{(j-1)}(J)) \geq D_j$.

Note that in an $(\eta, \boldsymbol{\delta}, \geq\boldsymbol{D}, \boldsymbol{a})$-family $\mathscr{P} = \{\mathscr{P}^{(1)}, \ldots, \mathscr{P}^{(k-1)}\}$ (cf. Definition 3.2), the vertices, pairs, $\ldots$, and $(k-1)$-tuples of V are under regular control. The following definition describes how the family $\mathscr{P}$ will control the edges of a hypergraph $\mathcal{H}^{(k)}$, where $V = V(\mathcal{H}^{(k)})$.

Definition 3.3 (($\mathcal{H}^{(k)}, \mathscr{P}$) has $\mathbf{DEV}(\delta_k)$) Let $\delta_k > 0$ be given. For a k-graph $\mathcal{H}^{(k)}$ and an $\boldsymbol{a}$-family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ on $V = V(\mathcal{H}^{(k)})$, we say $(\mathcal{H}^{(k)}, \mathscr{P})$ has $\mathbf{DEV}(\delta_k)$ if

$$\Big| \bigcup \big\{ \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \colon \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \text{ satisfies that } (\mathcal{H}^{(k)}, \hat{\mathcal{P}}^{(k-1)}) \text{ does not have } \mathbf{DEV}\big(d(\mathcal{H}^{(k)} | \hat{\mathcal{P}}^{(k-1)}), \delta_k\big) \big\} \Big| \leq \delta_k |V|^k.$$

Before we state the algorithmic hypergraph regularity lemma, we say a word about some notation we use in it.

Remark 3.4 Let $\boldsymbol{D} = (D_2, \ldots, D_{k-1}) \in (0,1]^{k-1}$ be a sequence, and for each $2 \leq i \leq k-1$, let $\delta_i : (0,1]^{k-i} \to (0,1)$ be a function (of $k-i$ many $(0,1]$ variables), where we write $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_{k-1})$. We shall use the notation

$$\boldsymbol{\delta}(\boldsymbol{D}) = (\delta_i(D_i, \ldots, D_{k-1}) : 2 \leq i \leq k-1)$$

to denote the sequence of function values whose i^{th} coordinate, $2 \leq i \leq k-1$, is $\delta_i(D_i, \ldots, D_{k-1})$. We consider this concept since, in most applications of Theorem 3.5, one needs the value δ_i to be sufficiently small not only w.r.t. D_i, but also $D_{i+1}, \ldots, D_{k-1}$. $\square$

We now state the algorithmic hypergraph regularity lemma.

Theorem 3.5 (Algorithmic HRL) *Let $k \geq 2$ be a fixed integer, and let $\eta, \delta_k > 0$ be fixed positives. For each $2 \leq i \leq k-1$, let $\delta_i : (0,1]^{k-i} \to (0,1)$ be a function, and set $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_{k-1})$. Then, there exist $t, n_0 \in \mathbb{N}$ so that the following holds.*

For every k-uniform hypergraph $\mathcal{H}^{(k)}$ with $|V(\mathcal{H}^{(k)})| = n \geq n_0$, one may construct, in time $O(n^{3k})$, a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ of $V(\mathcal{H}^{(k)})$ with the following properties:

(i) *$\mathscr{P}$ is a t-bounded $(\eta, \boldsymbol{\delta}(\boldsymbol{D}), \geq\boldsymbol{D}, \boldsymbol{a}^{\mathscr{P}})$-family on $V(\mathcal{H}^{(k)})$ (cf. Remark 3.4);*

(ii) *$(\mathcal{H}^{(k)}, \mathscr{P})$ has $\mathbf{DEV}(\delta_k)$.*

We proceed with the following remark.

Remark 3.6 Similarly as in Szemerédi [22, 23] for graphs, it is well-known that one can prove a hypergraph regularity lemma which 'regularizes' not one, but multiple hypergraphs $\mathcal{H}_1^{(k)}, \ldots, \mathcal{H}_s^{(k)}$ (on a common vertex set V) simultaneously. More precisely, in the context of Theorem 3.5, the t-bounded $(\eta, \boldsymbol{\delta}(\boldsymbol{D}), \geq\boldsymbol{D}, \boldsymbol{a}^{\mathscr{P}})$-family above will satisfy that, for each $1 \leq i \leq s$, the pair $(\mathcal{H}_i^{(k)}, \mathscr{P})$ has $\mathbf{DEV}(\delta_k)$, where $t = t(s, k, \eta, \delta_k, \boldsymbol{\delta})$ and $|V| \geq n_0 = n_0(s, k, \eta, \delta_k, \boldsymbol{\delta})$.

We shall prove Theorem 3.5 by induction on $k \geq 2$. To avoid formalism, we shall be proving the case $s = 1$, but our induction hypothesis will assume the general case. $\square$

4 Proof of Theorem 3.5

The proof of Theorem 3.5 is by induction on $k \geq 2$. The induction begins with $k = 2$ as a known base case. Indeed, Alon et al. [1] proved an algorithmic version of the Szemerédi Regularity Lemma, which is Theorem 3.5 ($k = 2$) with **DEV** replaced by **DISC**. Gowers [4, 5] proved that **DEV** and **DISC** are equivalent properties when $k = 2$, and so the base case of Theorem 3.5 holds. We assume Theorem 3.5 holds through $k - 1 \geq 2$, and prove it for $k \geq 3$. To that end, we need a few supporting considerations.

Suppose $\mathcal{H}^{(k)}$ is a k-uniform hypergraph with vertex set $V = V(\mathcal{H}^{(k)})$, where $|V| = n$. Let $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ be an $\boldsymbol{a}$-family of partitions on V. We define the *index* of $\mathscr{P}$ w.r.t. $\mathcal{H}^{(k)}$ as

$$\begin{aligned}&\mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P})\\ &\quad = \frac{1}{n^k} \sum \Big\{ d^2(\mathcal{H}^{(k)} | \hat{\mathcal{P}}^{(k-1)}) \big| \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \big| : \\ &\qquad\qquad \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \Big\}.\end{aligned}$$

Clearly,

$$0 \leq \mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}) \leq 1. \tag{4}$$

The proof of Theorem 3.5 is similar to that of Szemerédi [22, 23], where we will use the following so-called *Index-pumping Lemma* (Lemma 4.1 below). To introduce this lemma, let $\mathcal{H}^{(k)}$ be a k-uniform hypergraph with vertex set $V = V(\mathcal{H}^{(k)})$, where $|V| = n$. Since this proof is by induction on k, suppose we already have a 'regular partition' $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ of V up through $k - 1$. More precisely,

- let $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ be an arbitrary t-bounded, $(\eta, \boldsymbol{\delta}(\boldsymbol{D}), \geq\!\boldsymbol{D}, \boldsymbol{a})$-family on V.

We now test how $\mathcal{H}^{(k)}$ behaves on $\mathscr{P}$. In particular, we test whether $(\mathcal{H}^{(k)}, \mathscr{P})$ has $\mathbf{DEV}(\delta_k)$, which we may do in time $O(n^{2k})$. Indeed,

- for each polyad $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$, we test (by using Definition 2.6) whether or not $(\mathcal{H}^{(k)}, \hat{\mathcal{P}}^{(k-1)})$ has $\mathbf{DEV}(d_{\hat{\mathcal{P}}^{(k-1)}}, \delta_k)$, where $d_{\hat{\mathcal{P}}^{(k-1)}} = d(\mathcal{H}^{(k)} | \hat{\mathcal{P}}^{(k-1)})$.

We arrive at two cases.

Case 1. Suppose we find that most polyads $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ satisfy that $(\mathcal{H}^{(k)}, \hat{\mathcal{P}}^{(k-1)})$ has $\mathbf{DEV}(d_{\hat{\mathcal{P}}^{(k-1)}}, \delta_k)$. Then we stop, and $\mathscr{P}$ is the partition we seek in Theorem 3.5.

Case 2. Suppose we find many polyads $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ for which the pair $(\mathcal{H}^{(k)}, \hat{\mathcal{P}}^{(k-1)})$ fails to have $\mathbf{DEV}(d_{\hat{\mathcal{P}}^{(k-1)}}, \delta_k)$. Then, for each such $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$,

- Theorem 2.12 builds (in time $O(n^{3k})$) an $r_{\hat{\mathcal{P}}^{(k-1)}}$-witness

$$\vec{\mathcal{Q}}^{(k-1)}_{\hat{\mathcal{P}}^{(k-1)}} = \{\mathcal{Q}^{(k-1)}_1, \ldots, \mathcal{Q}^{(k-1)}_{r_{\hat{\mathcal{P}}^{(k-1)}}}\}$$

of $\neg\,\mathbf{DISC}(d_{\hat{\mathcal{P}}^{(k-1)}}, \widetilde{\delta}_k, r_{\hat{\mathcal{P}}^{(k-1)}})$,

where $\widetilde{\delta}_k = \widetilde{\delta}_k(\delta_k) > 0$ depends on δ_k, and where $r_{\hat{\mathcal{P}}^{(k-1)}} \leq r(\boldsymbol{D})$, where $r(\boldsymbol{D})$ depends on $\boldsymbol{D}$. Now,

- Lemma 4.1 (below) constructs, in time $O(n^{k-1})$, a new partition $\mathscr{P}'$ from $\mathscr{P}$ and the witnesses $\vec{\mathcal{Q}}^{(k-1)}_{\hat{\mathcal{P}}^{(k-1)}}$, over those polyads $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ failing to have $\mathbf{DEV}(d_{\hat{\mathcal{P}}^{(k-1)}}, \delta_k)$, where

$$\mathrm{ind}_{\mathcal{H}^{(k)}}(\hat{\mathscr{P}}') \geq \mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}) + \frac{\widetilde{\delta}_k^4}{2}.$$

We now state the Index-pumping Lemma precisely.

Lemma 4.1 (Index-pumping Lemma) *Fix an integer $k \geq 2$, and let $\nu, \widetilde{\delta}_k > 0$ be fixed. For each $2 \leq i \leq k-1$, let $\delta_i : (0,1]^{k-i} \to (0,1)$ be a function, where we set $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_{k-1})$. Let $r : (0,1]^{k-2} \to \mathbb{N}$ be an arbitrary function. Let $\boldsymbol{D}_{\mathrm{old}} = (D_2^{\mathrm{old}}, \ldots, D_{k-1}^{\mathrm{old}}) \in (0,1]^{k-2}$ and $\boldsymbol{a}_{\mathrm{old}} = (a_1^{\mathrm{old}}, \ldots, a_{k-1}^{\mathrm{old}}) \in \mathbb{N}^{k-1}$ be fixed. Then, there exist $\boldsymbol{D}_{\mathrm{new}} = (D_2^{\mathrm{new}}, \ldots, D_{k-1}^{\mathrm{new}}) \in (0,1]^{k-2}$, $\boldsymbol{a}_{\mathrm{new}} = (a_1^{\mathrm{new}}, \ldots, a_{k-1}^{\mathrm{new}}) \in \mathbb{N}^{k-1}$, and $n_0 \in \mathbb{N}$ so that the following holds.*

Suppose $\mathcal{H}^{(k)}$ is a k-uniform hypergraph with vertex set $V = V(\mathcal{H}^{(k)})$, where $|V| = n \geq n_0$. Suppose $\mathscr{P}_{\mathrm{old}} = \mathscr{P}_{\mathrm{old}}(k-1, \boldsymbol{a})$ is a t_{old}-bounded $(\nu, \boldsymbol{\delta}(\boldsymbol{D}_{\mathrm{old}}), \geq\!\boldsymbol{D}_{\mathrm{old}}, \boldsymbol{a}_{\mathrm{old}})$-family on V, where $t_{\mathrm{old}} = \max\{a_1^{\mathrm{old}}, \ldots, a_{k-1}^{\mathrm{old}}\}$ and where $\boldsymbol{\delta}(\boldsymbol{D}_{\mathrm{old}}) = \big(\delta_i(D_i^{\mathrm{old}}, \ldots, D_{k-1}^{\mathrm{old}})\big)_{i=2}^{k-1}$. Suppose that $\hat{\mathscr{P}}^{(k-1)}_* \subseteq \hat{\mathscr{P}}^{(k-1)}$ is a given collection of polyads satisfying the following properties:

1. *$\forall\, \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_*$, one is given an $r_{\hat{\mathcal{P}}^{(k-1)}}$-witness $\vec{\mathcal{Q}}^{(k-1)}_{\hat{\mathcal{P}}^{(k-1)}}$ of $\neg\,\mathbf{DISC}(d_{\hat{\mathcal{P}}^{(k-1)}}, \widetilde{\delta}_k, r_{\hat{\mathcal{P}}^{(k-1)}})$, where $r_{\hat{\mathscr{P}}^{(k-1)}} \leq r(\boldsymbol{D}_{\mathrm{old}}) = r(D_2^{\mathrm{old}}, \ldots, D_{k-1}^{\mathrm{old}})$;*

2. $$\sum \Big\{ \big| \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \big| : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_* \Big\} \geq \widetilde{\delta}_k n^k.$$

Then,

(*a*) *there exists a* $t_{\rm new}$*-bounded* $(\nu, \boldsymbol{\delta}(\boldsymbol{D}_{\rm new}), \geq\boldsymbol{D}_{\rm new}, \boldsymbol{a}_{\rm new})$*-family* $\mathscr{P}_{\rm new} = \mathscr{P}_{\rm new}(k-1, \boldsymbol{a}_{\rm new})$ *on* V *for which*

$$\mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}_{\rm new}) \geq \mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}_{\rm old}) + \frac{\widetilde{\delta}_k^4}{2},$$

where $t_{\rm new} = \max\{a_1^{\rm new}, \dots, a_{k-1}^{\rm new}\}$ *and where* $\boldsymbol{\delta}(\boldsymbol{D}_{\rm new}) = \left(\delta_i(D_i^{\rm new}, \dots, D_{k-1}^{\rm new})\right)_{i=2}^{k-1}$.

(*b*) *Moreover, there exists an algorithm which, in time* $O(n^{k-1})$*, constructs the partition* $\mathscr{P}_{\rm new}$ *above from* $\mathscr{P}_{\rm old}$ *and the given collection of witnesses* $\{\vec{\mathcal{Q}}^{(k-1)}_{\hat{\mathcal{P}}^{(k-1)}} : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_*\}$.

Lemma 4.1 is essentially given as Lemma 8.3 of [19] and Lemma 6.3 of [5]. The proof of Lemma 4.1 is given in [5, 19], but with no focus to being algorithmic. We shall not give a formal proof of Lemma 4.1, but we will sketch a proof to indicate how its algorithmic part is obtained.

Indeed, the approach in [19] is similar to Szemerédi's [22, 23]. Consider the Venn Diagram of the intersections of the $r_{\hat{\mathcal{P}}^{(k-1)}}$-witnesses $\vec{\mathcal{Q}}^{(k-1)}_{\hat{\mathcal{P}}^{(k-1)}}$, over $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_*$. By Statement (1) in the hypothesis of Lemma 4.1, these witnesses are given to us. (In [19], these witnesses are assumed to exist, but here, we will build them with Theorem 2.12.) This Venn diagram has at most

$$2^{|\hat{\mathscr{P}}^{(k-1)}_*| r(\boldsymbol{D}_{\rm old})}$$

regions (this number is independent of n), where each region is a $(k-1, k-1)$-cylinder. This Venn Diagram defines a refinement $\mathscr{P}'_{\rm old}$ of $\mathscr{P}_{\rm old}$, so that $\mathscr{P}'_{\rm old}$ is itself a partition. The index of $\mathscr{P}'_{\rm old}$ will be larger than that of $\mathscr{P}_{\rm old}$ on account of the fact that, in Statement (2), we assumed many k-tuples were lost to polyads $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_*$. The $(k-1, k-1)$-cylinders of $\mathscr{P}'_{\rm old}$ may not have $\mathbf{DEV}(\delta_k)$, so we apply Theorem 3.5 to each (where we assume, by induction on k, that Theorem 3.5 is algorithmic for $k-1$ (cf. Remark 3.6)). This process produces the partition $\mathscr{P}_{\rm new}$, where it is well-known that, as a refinement of $\mathscr{P}'_{\rm old}$, we have $\mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}_{\rm new}) \geq \mathrm{ind}_{\mathcal{H}^{(k)}}(\mathscr{P}'_{\rm old})$. For the formal details of this outline, see [5, 19].

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