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Hereditary Properties of Triple Systems

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For an integer $s \geq 2$, a property $\Pi^{(s)}$ is an infinite class of s-uniform hypergraphs closed under isomorphism. We say that a property $\Pi^{(s)}$ is *hereditary* if $\Pi^{(s)}$ is closed under taking induced subhypergraphs. Thus, for some 'forbidden class' $\mathbb{F} = \{\mathcal{F}_i^{(s)}: i \in I\}$ of s-uniform hypergraphs, $\Pi^{(s)}$ is the set of all s-uniform hypergraphs not containing any $\mathcal{F}_i^{(s)} \in \mathbb{F}$ as an induced subhypergraph. Let $\Pi_n^{(s)}$ be those hypergraphs of $\Pi^{(s)}$ on some fixed n-vertex set. For a set of s-uniform hypergraphs $\mathbb{F} = \{\mathcal{F}_i^{(s)}: i \in I\}$, let

$$\mathrm{ex}_{\mathrm{ind}}(n, \mathbb{F}) = \max \left| [n]^s \setminus (\mathcal{M} \cup \mathcal{N}) \right| ,$$

where the maximum is taken over all $\mathcal{M}$ and $\mathcal{N} \subseteq [n]^s$ with $\mathcal{M} \cap \mathcal{N} = \emptyset$ such that, for all $\mathcal{G} \subseteq [n]^s \setminus (\mathcal{M} \cup \mathcal{N})$, no $\mathcal{F}_i^{(s)} \in \mathbb{F}$ appears as an induced subhypergraph of $\mathcal{G} \cup \mathcal{M}$. We show that for $s = 3$ and any hereditary property $\Pi^{(3)}$,

$$\log_2 |\Pi_n^{(3)}| = \mathrm{ex}_{\mathrm{ind}}(n, \mathbb{F}) + o(n^3)$$

holds, where $\mathbb{F}$ is a forbidden class associated with $\Pi^{(3)}$. This result complements a collection of analogous theorems already proven for graphs (i.e., $s = 2$).

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1. Introduction

Our aim in this paper is to investigate the asymptotic rate of growth of hereditary families of hypergraphs as we let the number of vertices of the hypergraphs tend to infinity. Before we proceed, we need to introduce some notation and some definitions.

Let n and s be natural numbers. As usual, let $[n]^s$ denote the set of all s-element subsets of $[n] = \{1, \dots, n\}$. More generally, if V is a set, we write $[V]^s$ for the set of all s-element subsets of V. An *s-uniform hypergraph*, or an *s-graph* for short, is a collection of s-element subsets, called *hyperedges* or simply *edges*, of a given *vertex set*. We often identify our hypergraphs with their edge sets. In particular, we often write $|\mathcal{F}|$ for the number of edges in a hypergraph $\mathcal{F}$. Our terminology concerning hypergraphs is standard; see, e.g., Bollobás [2].

For an integer $s \geq 2$, a *property* $\Pi^{(s)}$ is an infinite class of s-uniform hypergraphs closed under isomorphism. We say that a property is *hereditary* provided it is closed under taking *induced subhypergraphs*, and we say that a property is *monotone* if it is closed under taking *arbitrary* subhypergraphs. Clearly, every monotone property is also hereditary.

Observe that, for every hereditary property $\Pi^{(s)}$, there exists a unique minimal family $\mathbb{F} = \{\mathcal{F}_i^{(s)} \colon i \in I\}$ of pairwise non-isomorphic s-uniform hypergraphs so that $\Pi^{(s)}$ is the set of all s-uniform hypergraphs not containing any $\mathcal{F}_i^{(s)} \in \mathbb{F}$ as an induced subhypergraph. We refer to this set of hypergraphs $\mathbb{F}$ for $\Pi^{(s)}$ as the *forbidden class* for $\Pi^{(s)}$. Similarly, every monotone property $\Pi^{(s)}$ admits a unique minimal forbidden class $\mathbb{H} = \{\mathcal{H}_i^{(s)} \colon i \in I\}$ of pairwise non-isomorphic hypergraphs so that $\Pi^{(s)}$ is the set of all s-uniform hypergraphs not containing any $\mathcal{H}^{(s)} \in \mathbb{H}$ as a subhypergraph.

Using forbidden classes, we may develop notation to describe hereditary properties $\Pi^{(s)}$ in an alternative manner. For a class of hypergraphs $\mathbb{F} = \{\mathcal{F}_i^{(s)} \colon i \in I\}$, denote by $\mathrm{Forb}_{\mathrm{ind}}(\mathbb{F})$ the class of all s-uniform hypergraphs containing no copy $\mathcal{F}_i^{(s)} \in \mathbb{F}$ as an induced subhypergraph. Then a hereditary property $\Pi^{(s)}$ may be equivalently written as $\mathrm{Forb}_{\mathrm{ind}}(\mathbb{F})$, where $\mathbb{F}$ is the forbidden class for $\Pi^{(s)}$. Similarly, denote by $\mathrm{Forb}(\mathbb{H})$ the class of all s-uniform hypergraphs containing no $\mathcal{H}^{(s)} \in \mathbb{H}$ as a subhypergraph. We may then write a monotone property $\Pi^{(s)}$ as $\mathrm{Forb}(\mathbb{H})$, where $\mathbb{H}$ is the forbidden class for $\Pi^{(s)}$.

For a hereditary property $\Pi^{(s)}$, we denote those hypergraphs of $\Pi^{(s)}$ on some fixed n-element vertex set by $\Pi_n^{(s)}$. Using the alternative notation for a hereditary property $\Pi^{(s)}$, we may denote those hypergraphs of $\mathrm{Forb}_{\mathrm{ind}}(\mathbb{F})$ on some fixed n-element vertex set by $\mathrm{Forb}_{\mathrm{ind}}(n, \mathbb{F})$. Similarly, we denote those hypergraphs of $\mathrm{Forb}(\mathbb{H})$ on some fixed n-element vertex set by $\mathrm{Forb}(n, \mathbb{H})$. In what follows, we are interested in the rate of growth of $|\Pi_n^{(s)}|$ for hereditary properties $\Pi^{(s)}$ as $n \to \infty$.

We now review some of the first results in this area, obtained for the case in which $s = 2$ and $\Pi^{(s)}$ is monotone.

Monotone properties of graphs. For a positive integer r, consider the monotone property $\Pi^{(2)} = \mathrm{Forb}(K_{r+1})$. Erdős, Kleitman, and Rothschild [8] showed that

$$\log_2 |\mathrm{Forb}(n, K_{r+1})| = \left(1 - \frac{1}{r} + o(1)\right) \binom{n}{2}.$$

This result of [8] was later extended by Erdős, Frankl, and Rödl [7] to say that if H is any graph with chromatic number $\chi(H) = r + 1$, then

$$\log_2 |\mathrm{Forb}(n, H)| = \left(1 - \frac{1}{r} + o(1)\right) \binom{n}{2}. \tag{1}$$

For a property $\Pi^{(s)}$, put $c_n(\Pi^{(s)}) = \binom{n}{s}^{-1} \log_2 |\Pi_n^{(s)}|$, and set $c(\Pi^{(s)}) = \lim_{n\to\infty} c_n(\Pi^{(s)})$, when the limit exists. Note that we may express (1) by saying that

$$c(\mathrm{Forb}(H)) = 1 - \frac{1}{\chi(H) - 1}.$$

It turns out that using the same proof from [7] for (1), one may easily determine $c(\Pi^{(2)})$ for any monotone property $\Pi^{(2)}$. Specifically, for a monotone property $\Pi^{(2)}$, let $\Pi^{(2)} = \mathrm{Forb}(\mathbb{F})$, where $\mathbb{F}$ is the forbidden class for $\Pi^{(2)}$. Set $r_0 = \min\{\chi(F)\colon F \in \mathbb{F}\}$. Then along the same lines as [7], one may show that

$$c(\Pi^{(2)}) = c(\mathrm{Forb}(\mathbb{F})) = 1 - \frac{1}{r_0 - 1}.$$

Let us now turn to *hereditary* properties $\Pi^{(2)}$ of graphs.

Hereditary properties of graphs. Studying $c(\Pi^{(2)})$ in the case in which $\Pi^{(2)}$ is a hereditary property is more difficult than in the case in which $\Pi^{(2)}$ is monotone. Prömel and Steger [17, 18, 19, 20] were the first to work on this problem, and studied, for an arbitrary graph H, the hereditary property $\mathrm{Forb}_{\mathrm{ind}}(H)$. They defined an integer-valued parameter $r(H)$ similar to but more sophisticated than the chromatic number $\chi(H)$, and proved that $c(\mathrm{Forb}_{\mathrm{ind}}(H)) = 1 - 1/(r(H) - 1)$.

Scheinermann and Zito [22] asked the more general question of whether $c(\Pi^{(2)})$ always exists for a hereditary property, and if so, what its possible values are. Alekseev [1] and Bollobás and Thomason [4] established the existence of $c(\Pi^{(2)})$ and determined its evaluation for *every* hereditary property $\Pi^{(2)}$. They defined an integer-valued parameter $r(\Pi^{(2)})$ by extending the definition for $r(H)$ given in [19], and proved that for any hereditary property $\Pi^{(2)}$, we have $c(\Pi^{(2)}) = 1 - 1/(r(\Pi^{(2)}) - 1)$.

In a very recent paper, Bollobás and Thomason [5] investigate the structure of hereditary properties in the space of random graphs $\mathcal{G}(n, p)$; their results substantially extend the ones in [1] and [4], which may be viewed as the $p = 1/2$ case of the more general results in [5].

We now turn to hypergraphs.

Monotone properties of hypergraphs. Recent work in [15] studied $c(\Pi^{(3)})$ for monotone properties $\Pi^{(3)}$ of triple systems. We recall the following well-known definition. For a set of 3-uniform hypergraphs $\mathbb{H} = \{\mathcal{H}_i^{(3)}\colon i \in I\}$ and an integer n, define the *Turán number* $\mathrm{ex}(n, \mathbb{H})$ as

$$\mathrm{ex}(n, \mathbb{H}) = \max\{|\mathcal{G}|\colon \mathcal{G} \in \mathrm{Forb}(n, \mathbb{H})\}. \tag{2}$$

A well-known averaging argument of Katona, Nemetz, and Simonovits [12] shows that, for every family of 3-uniform hypergraphs $\mathbb{H}$, the limiting Turán ratio

$$\mathrm{ex}(\mathbb{H}) = \lim_{n\to\infty} \binom{n}{3}^{-1} \mathrm{ex}(n, \mathbb{H}) \tag{3}$$

exists. However, determining this limit, even in the case that $\mathbb{H}$ consists of a single hypergraph (say, $\mathbb{H} = \{K_4^{(3)}\}$, where $K_4^{(3)}$ is the complete triple system on 4 vertices), is a notoriously hard problem, central to extremal combinatorics. In fact, very few such numbers are known.

The work of [15] connected the problem of determining $c(\Pi^{(3)})$ for monotone properties $\Pi^{(3)}$ to determining Turán numbers for families of triple systems. For a monotone property $\Pi^{(3)}$, write $\Pi^{(3)} = \mathrm{Forb}(\mathbb{H})$ where $\mathbb{H}$ is the forbidden class for $\Pi^{(3)}$. In [15], it was shown that

$$c(\Pi^{(3)}) = c(\mathrm{Forb}(\mathbb{H})) = \mathrm{ex}(\mathbb{H}). \tag{4}$$

The equality in (4) may be viewed as an extension of (1). Indeed, for a family of graphs $\mathbb{H}$, let $\mathrm{ex}(n, \mathbb{H})$ be defined analogously to (2) and set $\mathrm{ex}(\mathbb{H}) = \lim_{n\to\infty} \binom{n}{2}^{-1} \mathrm{ex}(n, \mathbb{H})$. The well-known Erdős–Stone theorem [9] states that $\mathrm{ex}(\mathbb{H}) = 1 - 1/(r_0 - 1)$, where $r_0 = \min\{\chi(H)\colon H \in \mathbb{H}\}$. Note that (4) does not give an explicit numerical evaluation of $c(\Pi^{(3)})$ for most monotone properties $\Pi^{(3)}$, as the corresponding task of determining $\mathrm{ex}(\mathbb{H})$ is very difficult for most families $\mathbb{H}$.

We now discuss *hereditary* properties of hypergraphs, and present the main result of this paper.

Hereditary properties of hypergraphs. Returning to the original question of Scheinermann and Zito, one may ask whether $c(\Pi^{(s)})$ exists for all hereditary properties $\Pi^{(s)}$ for arbitrary $s \geq 3$. Bollobás and Thomason [3] proved that this is indeed is the case. Their proof, which is based on a beautiful volume inequality extending the Loomis–Whitney inequality [14], gives, however, no indication as to the possible values of $c(\Pi^{(s)})$.

In this paper, we study $c(\Pi^{(3)})$ for hereditary properties $\Pi^{(3)}$ of 3-uniform hypergraphs, and prove an identity similar to (4) above. To that end, we consider the following definition extended from [19]. Let $\mathbb{F} = \{\mathcal{F}_i^{(3)}\colon i \in I\}$ be a fixed class of 3-uniform hypergraphs. For an integer n, consider sets $\mathcal{M}$ and $\mathcal{N} \subseteq [n]^3$ with the following two properties:

(*i*) $\mathcal{M} \cap \mathcal{N} = \emptyset$.

(*ii*) For all $\mathcal{G} \subseteq [n]^3 \setminus (\mathcal{M} \cup \mathcal{N})$, we have $\mathcal{G} \cup \mathcal{M} \in \mathrm{Forb}_{\mathrm{ind}}(n, \mathbb{F})$.

Let

$$\mathrm{ex}_{\mathrm{ind}}(n, \mathbb{F}) = \max \left|[n]^3 \setminus (\mathcal{M} \cup \mathcal{N})\right|,$$

where the maximum is taken over all $\mathcal{M}$ and $\mathcal{N}$ that have properties (*i*) and (*ii*) above. It is not difficult to show, again by averaging, that for all families of 3-uniform hypergraphs $\mathbb{F}$, the sequence $\binom{n}{3}^{-1} \mathrm{ex}_{\mathrm{ind}}(n, \mathbb{F})$ $(n \geq 3)$ is non-increasing, and hence

$$\mathrm{ex}_{\mathrm{ind}}(\mathbb{F}) = \lim_{n\to\infty} \binom{n}{3}^{-1} \mathrm{ex}_{\mathrm{ind}}(n, \mathbb{F}) \tag{5}$$

exists. Our result is then as follows.

Theorem 1. *For any hereditary property $\Pi^{(3)}$ of triple systems,*

$$\log_2 |\Pi_n^{(3)}| = \mathrm{ex}_{\mathrm{ind}}(n, \mathbb{F}) + o(n^3),$$

where $\mathbb{F}$ is the forbidden class of triple systems for $\Pi^{(3)}$. In particular,

$$c(\Pi^{(3)}) = \mathrm{ex}_{\mathrm{ind}}(\mathbb{F}).$$

As mentioned with (4), determining extremal limits for families of hypergraphs is very difficult. We expect determining $\mathrm{ex}_{\mathrm{ind}}(\mathbb{F})$ to be at least as difficult. In fact, we show in the next section that there are some hypergraphs $\mathcal{F}^{(3)}$ for which determining $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(3)})$ is equivalent to determining $\mathrm{ex}(n, \mathcal{F}^{(3)})$.

The rest of this paper is organized as follows. In the next section, we give some background facts concerning the parameter $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)})$ for single hypergraphs $\mathcal{F}^{(s)}$. In Section 3, we present the main tools of our proof for Theorem 1, the Regularity Lemma for triple systems of [11] and the Counting Lemma of [16]. In Section 4, we give the proof of Theorem 1.

2. The parameter $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)})$ for single s-uniform hypergraphs $\mathcal{F}^{(s)}$

We begin this section with the following definitions. Let n be an integer and let $\mathcal{F}^{(s)}$ be an s-uniform hypergraph. Let $\mathcal{M}$, $\mathcal{N} \subseteq [n]^s$ have the following two properties:

(i) $\mathcal{M} \cap \mathcal{N} = \emptyset$.

(ii) For all $\mathcal{G}' \subseteq \mathcal{G} = [n]^s \setminus (\mathcal{M} \cup \mathcal{N})$, we have $\mathcal{G}' \cup \mathcal{M} \in \mathrm{Forb}_{\mathrm{ind}}(n, \mathcal{F}^{(s)})$, that is, the hypergraph $\mathcal{F}^{(s)}$ is not an induced subhypergraph of $\mathcal{G}' \cup \mathcal{M}$.

It may be worth noting that, in (ii) above, $[n]^s = \mathcal{G} \cup \mathcal{M} \cup \mathcal{N}$ is a partition of $[n]^s$. Define

$$\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)}) = \max |\mathcal{G}| = \max \big|[n]^s \setminus (\mathcal{M} \cup \mathcal{N})\big|,$$

where the maximum is taken over all $\mathcal{M}$ and $\mathcal{N}$ that have properties (i) and (ii) above. Note that $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)})$ is the special case of $\mathrm{ex}_{\mathrm{ind}}(n, \mathbb{F})$, defined in the abstract, when the family $\mathbb{F}$ consists of the single hypergraph $\mathcal{F}^{(s)}$. Analogously to (5), it is routine to show that, for all hypergraphs $\mathcal{F}^{(s)}$, the limit

$$\mathrm{ex}_{\mathrm{ind}}(\mathcal{F}^{(s)}) = \lim_{n \to \infty} \binom{n}{s}^{-1} \mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)})$$

exists.

For an integer n and an s-uniform hypergraph $\mathcal{F}^{(s)}$, let

$$\mathrm{ex}(n, \mathcal{F}^{(s)}) = \max\{|\mathcal{G}| \colon \mathcal{G} \in \mathrm{Forb}(n, \mathcal{F}^{(s)})\}.$$

Analogously to (3), it is not hard to show that, for all hypergraphs $\mathcal{F}^{(s)}$, the limit

$$\mathrm{ex}(\mathcal{F}^{(s)}) = \lim_{n \to \infty} \binom{n}{s}^{-1} \mathrm{ex}(n, \mathcal{F}^{(s)})$$

exists.

In this section, we present some properties of the function $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)})$ for s-uniform hypergraphs $\mathcal{F}^{(s)}$. All of the following facts are quite straightforward to prove, and therefore we only sketch the proofs of a few. In all that follows, $\mathcal{F}^{(s)}$ denotes an arbitrary s-uniform hypergraph. We begin with the following first fact.

Fact 2. $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)}) \geq \mathrm{ex}(n, \mathcal{F}^{(s)})$

Indeed, let $\mathcal{H}^{(s)}$ be an $\mathcal{F}^{(s)}$-free s-uniform hypergraph on n vertices and $\mathrm{ex}(n, \mathcal{F}^{(s)})$ edges. In the definition of $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)})$, set $\mathcal{G} = \mathcal{H}^{(s)}$, $\mathcal{M} = \emptyset$, and $\mathcal{N} = [n]^s \setminus \mathcal{G}$. Note that, as defined, $\mathcal{G}$, $\mathcal{M}$, and $\mathcal{N}$ satisfy properties (i) and (ii) from the definition of $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)})$. By definition, $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)}) \geq |\mathcal{G}| = \mathrm{ex}(n, \mathcal{F}^{(s)})$, and so the proof is complete.

In what follows, we denote the complete s-uniform hypergraph on k vertices by $K_k^{(s)}$. The next fact follows immediately from the definition of $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)})$.

Fact 3. $\mathrm{ex}_{\mathrm{ind}}(n, K_k^{(s)}) = \mathrm{ex}(n, K_k^{(s)})$

Denote by $K_k^{(s)} - e$ the s-uniform hypergraph of $\binom{k}{s} - 1$ edges on k points. The following fact, independently observed by D. Mubayi, complements Fact 3.

Fact 4. $\mathrm{ex}_{\mathrm{ind}}(n, K_k^{(s)} - e) = \mathrm{ex}(n, K_k^{(s)} - e)$

The proof of Fact 4 follows along the lines of the proof of upcoming Fact 7, so we omit the details here.

In what follows, we denote the complement $[V(\mathcal{F}^{(s)})]^s \setminus \mathcal{F}^{(s)}$ of an s-uniform hypergraph $\mathcal{F}^{(s)}$ by $\overline{\mathcal{F}^{(s)}}$.

Fact 5. $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)}) = \mathrm{ex}_{\mathrm{ind}}(n, \overline{\mathcal{F}^{(s)}})$

Indeed, if the partition $[n]^s = \mathcal{G} \cup \mathcal{M} \cup \mathcal{N}$ of $[n]^s$ establishes the value of $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)})$, then the partition $\underline{[n]^s} = \mathcal{G} \cup \mathcal{M}' \cup \mathcal{N}'$, where $\mathcal{M}' = \mathcal{N}$ and $\mathcal{N}' = \mathcal{M}$, establishes a lower bound for $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)})$, and vice-versa.

Facts 2 and 5 immediately imply the following.

Fact 6. $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)}) \geq \max\{\mathrm{ex}(n, \mathcal{F}^{(s)}), \mathrm{ex}(n, \overline{\mathcal{F}^{(s)}})\}$

We continue with the following fact.

Fact 7. *Let $\mathcal{F}_{s+1}^{(s)}$ be an s-uniform hypergraph with $s+1$ vertices and at least $(s+1)/2$ edges. Then $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}_{s+1}^{(s)}) = \mathrm{ex}(n, \mathcal{F}_{s+1}^{(s)})$.*

Indeed, let $\mathcal{F}_{s+1}^{(s)}$ have f edges, where $(s+1)/2 \leq f \leq s+1$. We mention in advance that the key observation in our proof of Fact 7 is that all s-uniform hypergraphs consisting of f edges on $s+1$ points are isomorphic.

Suppose, on the contrary, that $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)}_{s+1}) > \mathrm{ex}(n, \mathcal{F}^{(s)}_{s+1})$. Let the partition $[n]^s = \mathcal{G} \cup \mathcal{M} \cup \mathcal{N}$ of $[n]^s$ establish the value $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)}_{s+1})$. As $|\mathcal{G}| > \mathrm{ex}(n, \mathcal{F}^{(s)}_{s+1})$, the hypergraph $\mathcal{G}$ must contain a copy of $\mathcal{F}^{(s)}_{s+1}$. Let X_0 denote the vertex set of a copy of $\mathcal{F}^{(s)}_{s+1}$ in $\mathcal{G}$. Let

$$g_0 = \big|[X_0]^s \cap \mathcal{G}\big|, \qquad m_0 = \big|[X_0]^s \cap \mathcal{M}\big|, \quad \text{and} \quad n_0 = \big|[X_0]^s \cap \mathcal{N}\big|.$$

Note that

$$g_0 + m_0 + n_0 = s + 1,$$

and thus

$$g_0 + m_0 \le s + 1.$$

As $[X_0]^s \cap \mathcal{G}$ contains a copy of $\mathcal{F}^{(s)}_{s+1}$, we have

$$g_0 \ge f.$$

Consequently,

$$f + m_0 \le g_0 + m_0 \le s + 1,$$

and with $f \ge (s+1)/2$, we see that

$$m_0 \le \frac{s+1}{2} \le f \le g.$$

Let $\mathcal{G}' \subseteq \mathcal{G}$ be obtained by deleting any m_0 edges from $[X_0]^s \cap \mathcal{G}$. As all s-uniform hypergraphs of f edges on $s+1$ points are isomorphic, it is easy to see that $\mathcal{G}' \cup \mathcal{M}$ contains an induced copy of $\mathcal{F}^{(s)}_{s+1}$. This, however, contradicts that the partition $[n]^s = \mathcal{G} \cup \mathcal{M} \cup \mathcal{N}$ of $[n]^s$ establishes the value $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)}_{s+1})$.

Fact 8. *Let $\mathcal{F}^{(s)}_{s+1}$ be an s-uniform hypergraph with $s+1$ vertices and at most $(s+1)/2$ edges. Then $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)}_{s+1}) = \mathrm{ex}(n, \overline{\mathcal{F}^{(s)}_{s+1}})$.*

Indeed, by Facts 5 and 7, we have $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)}_{s+1}) = \mathrm{ex}_{\mathrm{ind}}(n, \overline{\mathcal{F}^{(s)}_{s+1}}) = \mathrm{ex}(n, \overline{\mathcal{F}^{(s)}_{s+1}})$.

For our next fact, we need the following notions. An s-uniform hypergraph $\mathcal{H}^{(s)}$ is called *t-partite* if there exists a partition $V(\mathcal{H}^{(s)}) = V_1 \cup \ldots \cup V_t$ so that every s-tuple of $\mathcal{H}^{(s)}$ meets each V_i $(1 \le i \le t)$ at most once. Denote by $K^{(s)}_{s+1} - 2e$ the s-uniform hypergraph consisting of $s-1$ edges on $s+1$ vertices. Observe that $K^{(3)}_4 - 2e$ is 3-partite, but $K^{(s)}_{s+1} - 2e$ is *not* s-partite for any $s \ge 4$. More generally, a hypergraph $\mathcal{F}^{(s)}_{s+1}(m)$ with $m \ge 3$ hyperedges is *not* s-partite for any $s \ge 2$. Consequently, for any $s \ge 3$, an s-uniform hypergraph $\mathcal{H}^{(s)}$ is such that both $\mathcal{H}^{(s)}$ and $\overline{\mathcal{H}^{(s)}}$ are s-partite if and only if $\mathcal{H}^{(s)}$ has at most s points, or else, $s = 3$ and $\mathcal{H}^{(3)} = K^{(3)}_4 - 2e$. An old theorem of P. Erdős [6] states that an s-uniform hypergraph $\mathcal{H}^{(s)}$ is s-partite if and only if $\mathrm{ex}(n, \mathcal{H}^{(s)}) = o(n^s)$. This theorem, together with Facts 2 and 6, implies the following.

Fact 9. *Suppose $s \ge 3$ and $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}^{(s)}) = o(n^s)$. Then $\mathcal{F}^{(s)}$ must either have at most s vertices, or else, $s = 3$ and $\mathcal{F}^{(3)} = K^{(3)}_4 - 2e$.*

We conclude this section with the following fact. Recall that for a fixed s-uniform hypergraph $\mathcal{F}^{(s)}$, we write $\mathrm{ex}_{\mathrm{ind}}(\mathcal{F}^{(s)})$ for the limit $\lim_{n\to\infty}\binom{n}{s}^{-1}\mathrm{ex}_{\mathrm{ind}}(n,\mathcal{F}^{(s)})$, which is always guaranteed to exist.

Fact 10. *The set $X = \{\mathrm{ex}_{\mathrm{ind}}(\mathcal{F}^{(s)})\colon \mathcal{F}^{(s)}\}$ is well-ordered.*

Indeed, we show that, (*) for every $\delta > 0$, the number of hypergraphs $\mathcal{F}^{(s)}$ with $0 \le \mathrm{ex}_{\mathrm{ind}}(\mathcal{F}^{(s)}) < 1-\delta$ is finite. To that end, we show that (**) there exists an integer $t = t(\delta)$ so that every $\mathcal{F}^{(s)}$ satisfying $0 \le \mathrm{ex}_{\mathrm{ind}}(\mathcal{F}^{(s)}) < 1-\delta$ also satisfies that both $\mathcal{F}^{(s)}$ and $\overline{\mathcal{F}^{(s)}}$ are t-partite. As there are only finitely many s-uniform hypergraphs $\mathcal{F}^{(s)}$ satisfying that both $\mathcal{F}^{(s)}$ and $\overline{\mathcal{F}^{(s)}}$ are t-partite (say, by Ramsey's theorem), assertion (*) follows and Fact 10 is proved.

We now prove assertion (**). Suppose to the contrary that there is $\delta > 0$ such that, for every integer t, there is a hypergraph $\mathcal{F}^{(s)}$ for which $\mathrm{ex}_{\mathrm{ind}}(\mathcal{F}^{(s)}) < 1-\delta$ but not both of $\mathcal{F}^{(s)}$ and $\overline{\mathcal{F}^{(s)}}$ are t-partite. Using Fact 5, we may assume, without loss of generality, that $\mathcal{F}^{(s)}$ is not t-partite. Moreover, we may suppose t to be sufficiently large for our inequalities to hold. For $n \ge n_0(t)$, let $m = n/t$, and consider the complete t-partite s-uniform hypergraph $K_t^{(s)}(m)$ on n vertices. As $\mathcal{F}^{(s)}$ is not t-partite, $\mathcal{F}^{(s)} \not\subseteq K_t^{(s)}(m)$; thus $\mathrm{ex}(n,\mathcal{F}^{(s)}) > \binom{t}{s}m^s > (1-\delta^2)\binom{n}{s}$. On the other hand, by Fact 2, $\mathrm{ex}_{\mathrm{ind}}(n,\mathcal{F}^{(s)}) \ge \mathrm{ex}(n,\mathcal{F}^{(s)}) > (1-\delta^2)\binom{n}{s}$, and, with n sufficiently large, $\binom{n}{s}^{-1}\mathrm{ex}_{\mathrm{ind}}(n,\mathcal{F}^{(s)})$ is less than, say, δ^2 apart from $\mathrm{ex}_{\mathrm{ind}}(\mathcal{F}^{(s)})$. Since $\mathrm{ex}_{\mathrm{ind}}(\mathcal{F}^{(s)}) < 1-\delta$, we have a contradiction for any small enough δ.

We close this section mentioning that the results in Frankl and Rödl [10] imply that the set

$$L_{<\infty} = \{\mathrm{ex}(\mathbb{F})\colon \mathbb{F} \text{ is a finite family of } s\text{-uniform hypergraphs}\}$$

is *not* well-ordered for any $s \ge 3$. As it turns out, the set $L_{<\infty}^{\mathrm{ind}}$, defined in the analogous way for the parameter $\mathrm{ex}_{\mathrm{ind}}(\mathbb{F})$, is not well ordered either. This shows that the set $L_{<\infty}^{\mathrm{ind}}$ is considerably richer than the set X defined in Fact 10.

3. The Regularity Lemma and Related Topics

In this section, we present the Regularity Lemma for triple systems of [11] and the Counting Lemma of [16]. The reader who is not familiar with the regularity lemma of Szemerédi [23] is strongly encouraged to study the excellent survey of Komlós and Simonovits [13], before reading this section.

We mention now that the remainder of this paper is concerned only with triple systems and graphs. As the distinction between graphs and triple systems will be clear in context, we suppress the superscript "(s)" in all that follows.

3.1. Graphs and Cylinders

In this subsection, we provide background definitions and notation used in [16].

Definition 11 (Partite cylinders, triads). We refer to a k-partite graph P with a fixed k-partition $(V_1,\ldots,V_k)$ of $V = V(P)$ as a *k-partite cylinder*, and write $P =$

$\bigcup_{1\le i<j\le k} P^{ij}$, where $P^{ij} = P[V_i, V_j] = \big\{\{v_i, v_j\} \in P\colon v_i \in V_i,\ v_j \in V_j\big\}$. If $B \in [k]^3$, then the 3-partite cylinder $P(B) = \bigcup_{\{i,j\}\in[B]^2} P^{ij}$ is referred to as a *triad*.

Definition 12 (Density $d_P(X,Y)$). Suppose $P \subseteq [V]^2$ is a graph with vertex set $V = V(P)$, and let $X,\ Y \subseteq V$ be two nonempty disjoint subsets of V. We define the *density* $d_P(X,Y)$ of the pair (X,Y) with respect to P as

$$d_P(X,Y) = \frac{|\{x,y\} \in P\colon x \in X,\ y \in Y\}|}{|X||Y|}.$$

Definition 13 ((l,ε,k)-cylinder). Suppose $P = \bigcup_{1\le i<j\le k} P^{ij}$ is a k-partite cylinder with k-partition $(V_1,\dots,V_k)$, and let a positive integer l and a real number $\varepsilon > 0$ be given. We call P an *(l,ε,k)-cylinder* provided all pairs (V_i,V_j), $1 \le i < j \le k$, induce P^{ij} satisfying that, whenever $V_i' \subseteq V_i$ and $V_j' \subseteq V_j$ are such that $|V_i'| > \varepsilon|V_i|$ and $|V_j'| > \varepsilon|V_j|$, we have

$$\frac{1}{l}(1-\varepsilon) < d_{P^{ij}}(V_i',V_j') < \frac{1}{l}(1+\varepsilon).$$

The artificial condition in Definition 13 that l should be an integer will become natural later, when we discuss the regularity lemma for hypergraphs in Section 3.3.

We now define an auxiliary set system pertaining to a k-partite cylinder P.

Definition 14 ($\mathcal{K}_j(P)$). For a k-partite cylinder P, we denote by $\mathcal{K}_j(P)$ $(1 \le j \le k)$ the j-uniform hypergraph whose edges are precisely those j-element subsets of $V(P)$ that span cliques of order j in P.

Note that the quantity $|\mathcal{K}_j(P)|$ is the total number of cliques in P of order j, that is, $\big|\mathcal{K}_j(P)\big| = \big|\{X \subseteq V(P)\colon |X| = j,\ [X]^2 \subseteq P\}\big|$. As it turns out, for an (l,ε,k)-cylinder P, the quantity $|\mathcal{K}_j(P)|$ is easy to estimate.

Fact 15. *For any positive integers k and l and a positive real θ, there exists $\varepsilon > 0$ such that, whenever P is an (l,ε,k)-cylinder with k-partition $(V_1,\dots,V_k)$, where $|V_1| = \dots = |V_k| = m$, we have*

$$(1-\theta)m^k l^{-\binom{k}{2}} < |\mathcal{K}_k(P)| < (1+\theta)m^k l^{-\binom{k}{2}}. \tag{6}$$

Fact 15 is by now standard; the reader may for instance consult [13, Theorem 2.1].

3.2. 3-Uniform Hypergraphs and 3-Cylinders

In this subsection, we give definitions pertaining to 3-uniform hypergraphs. Recall that by a 3-uniform hypergraph $\mathcal{H}$ on vertex set V, we mean $\mathcal{H} \subseteq [V]^3$. As mentioned before, when there is no confusion in doing so, we identify the hypergraph $\mathcal{H}$ with its set of triples.

By a *k-partite* 3-uniform hypergraph $\mathcal{H}$ with a k-partition $(V_1,\dots,V_k)$, we understand a hypergraph $\mathcal{H}$ with its vertex set $V = V(\mathcal{H})$ partitioned into k classes $V = V_1 \cup \dots \cup V_k$, where each triple $e \in \mathcal{H}$ satisfies that for each $i \in [k]$, we have $|e \cap V_i| \le 1$.

Definition 16 (k-partite 3-cylinder; $\mathcal{H}(B)$). We refer to any k-partite, 3-uniform hypergraph $\mathcal{H}$ with a fixed k-partition $(V_1, \dots, V_k)$ as a *k-partite 3-cylinder.* For $B \in [k]^3$, we define $\mathcal{H}(B)$ as that 3-partite 3-cylinder induced by $\mathcal{H}$ on $\bigcup_{i \in B} V_i$.

Definition 17 (P underlies $\mathcal{H}$). Suppose P is a k-partite cylinder with k-partition $(V_1, \dots, V_k)$, and $\mathcal{H}$ is a k-partite 3-cylinder with k-partition $(V_1, \dots, V_k)$. We say that P *underlies* the 3-cylinder $\mathcal{H}$ if $\mathcal{H} \subseteq \mathcal{K}_3(P)$.

As in Definition 14, we define an auxiliary set system pertaining to the 3-cylinder $\mathcal{H}$.

Definition 18 ($\mathcal{K}_j(\mathcal{H})$). For a k-partite 3-cylinder $\mathcal{H}$, denote by $\mathcal{K}_j(\mathcal{H})$ $(1 \le j \le k)$, that j-uniform hypergraph whose edges are precisely those j-element subsets of $V(\mathcal{H})$ that span a clique of order j in $\mathcal{H}$.

Clearly, the quantity $|\mathcal{K}_j(\mathcal{H})|$ defined above is the total number of cliques of order j in $\mathcal{H}$, that is, $\big|\mathcal{K}_j(\mathcal{H})\big| = \big|\{X \subseteq V(\mathcal{H})\colon |X| = j,\ [X]^3 \subseteq \mathcal{H}\}\big|$. We now define a notion of density for hypergraphs.

Definition 19 (Density $d_{\mathcal{H}}(P)$). Let $\mathcal{H}$ be a 3-partite 3-cylinder and suppose $P = P^{12} \cup P^{23} \cup P^{13}$ is a 3-partite cylinder that underlies $\mathcal{H}$. We define the *density $d_{\mathcal{H}}(P)$ of $\mathcal{H}$ with respect to P* as

$$d_{\mathcal{H}}(P) = \begin{cases} |\mathcal{H}|/|\mathcal{K}_3(P)| & \text{if } |\mathcal{K}_3(P)| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that we may think of the notion of density introduced above as a measure of the proportion of triangles of P that are also triples of $\mathcal{H}$. More generally, we shall have to measure the density of our system $\mathcal{H}$ against 'thinner' triads. Suppose $Q \subseteq P$, where $Q = \bigcup_{\{i,j\} \in [3]^2} Q^{ij}$ and $Q^{ij} \subseteq P^{ij}$ $(1 \le i < j \le 3)$. One can define the density $d_{\mathcal{H}}(Q)$ of $\mathcal{H}$ with respect to Q as

$$d_{\mathcal{H}}(Q) = \begin{cases} |\mathcal{H} \cap \mathcal{K}_3(Q)|/|\mathcal{K}_3(Q)| & \text{if } |\mathcal{K}_3(Q)| > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

We shall need an extension of the definition in (7) above: we shall consider a 'simultaneous density' of $\mathcal{H}$ with respect to a fixed r-tuple of triads $(Q(1), \dots, Q(r))$.

Definition 20 (Density $d_{\mathcal{H}}(\vec{Q})$). Let $\mathcal{H}$ be a 3-partite 3-cylinder with an underlying 3-partite cylinder $P = P^{12} \cup P^{23} \cup P^{13}$. Let $\vec{Q} = (Q(1), \dots, Q(r))$ be an r-tuple of triads $Q(s) = \bigcup_{\{i,j\} \in [3]^2} Q^{ij}(s)$ satisfying that, for every $s \in [r]$ and every $\{i,j\} \in [3]^2$, we have $Q^{ij}(s) \subseteq P^{ij}$. We define the *density $d_{\mathcal{H}}(\vec{Q})$ of $\mathcal{H}$ with respect to $\vec{Q}$* as

$$d_{\mathcal{H}}(\vec{Q}) = \begin{cases} \Big|\mathcal{H} \cap \bigcup_{s=1}^r \mathcal{K}_3(Q(s))\Big| \Big/ \Big|\bigcup_{s=1}^r \mathcal{K}_3(Q(s))\Big| & \text{if } |\bigcup_{s=1}^r \mathcal{K}_3(Q(s))| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We now give a definition that provides a notion of regularity for 3-cylinders.

Definition 21 (**(δ, r)- and (α, δ, r)-regularity**). Let $\mathcal{H}$ be a 3-partite 3-cylinder with an underlying 3-partite cylinder $P = P^{12} \cup P^{23} \cup P^{13}$. Let r be a given positive integer and let $\delta > 0$ be given. We say that $\mathcal{H}$ is *(δ, r)-regular with respect to P* if condition (*) below is satisfied.

(*) Let $\vec{Q} = (Q(1), \dots, Q(r))$ be an arbitrary r-tuple of triads $Q(s) = \bigcup_{\{i,j\} \in [3]^2} Q^{ij}(s)$, where $Q^{ij}(s) \subseteq P^{ij}$ for all $s \in [r]$ and $\{i, j\} \in [3]^2$. Then if

$$\Big| \bigcup_{s=1}^{r} \mathcal{K}_3(Q(s)) \Big| > \delta \big| \mathcal{K}_3(P) \big|,$$

we have

$$|d_{\mathcal{H}}(\vec{Q}) - d_{\mathcal{H}}(P)| < \delta. \tag{8}$$

If, moreover, it is specified that $\mathcal{H}$ is (δ, r)-regular with respect to P with density $d_{\mathcal{H}}(P) = \alpha$ for some α, then we say that $\mathcal{H}$ is *(α, δ, r)-regular with respect to P*. If $\mathcal{H}$ is not (δ, r)-regular with respect to P, then we say that $\mathcal{H}$ is *(δ, r)-irregular* with respect to P.

3.3. The Regularity Lemma

In this section, we state a regularity lemma for 3-uniform hypergraphs established in [11]. First we state a number of supporting definitions.

Definition 22 (**Equitable partition; $m = |V_i|$**). Let t be an integer and let V be an n-element set. We define an *equitable partition* of V as a partition $V = V_0 \cup V_1 \cup \ldots \cup V_t$ of V, where

(*i*) $|V_1| = \ldots = |V_t| = \lfloor n/t \rfloor = m$,
(*ii*) $|V_0| < t$.

Definition 23 (**$(l, t, \gamma, \varepsilon)$-partition**). Let V be a set, γ and ε be given positive constants, and l and t be given positive integers. An *$(l, t, \gamma, \varepsilon)$-partition* $\mathbf{P}$ of $[V]^2$ is an (auxiliary) partition $V = V_0 \cup \ldots \cup V_t$ of V, together with a system of edge-disjoint bipartite graphs P^{ij}_α $(1 \le i < j \le t,\ \alpha = 0, 1, \ldots, l_{ij} \le l)$ such that

(*i*) $V = V_0 \cup \ldots \cup V_t$ is an equitable partition of V,
(*ii*) $\bigcup_{\alpha=0}^{l_{ij}} P^{ij}_\alpha = K(V_i, V_j)$ for all $1 \le i < j \le t$; in other words, the bipartite graphs P^{ij}_α $(0 \le \alpha \le l_{ij})$ form a decomposition of the complete bipartite graph $K(V_i, V_j)$ with vertex sets V_i and V_j,
(*iii*) all but $\le \gamma \binom{t}{2} m^2$ pairs $\{v_i, v_j\}$ with $v_i \in V_i$ and $v_j \in V_j$, where $1 \le i < j \le t$, are edges of $\varepsilon/2l$-regular bipartite graphs P^{ij}_α, and
(*iv*) for all but $\le \gamma \binom{t}{2}$ pairs (i, j), where $1 \le i < j \le t$, we have $\big|P^{ij}_0\big| \le \gamma m^2$ and

$$\frac{1}{l}\left(1 - \frac{\varepsilon}{2}\right) \le d_{P^{ij}_\alpha}(V_i, V_j) \le \frac{1}{l}\left(1 + \frac{\varepsilon}{2}\right)$$

for all $\alpha = 1, \ldots, l_{ij}$.

Note that by (*iii*) and (*iv*) above, we are ensured that nearly all P^{ij}_α are $(l, \varepsilon, 2)$-cylinders (see Fact 26 below).

Definition 24 ((δ, r)-regular partition P). Suppose we have a 3-uniform hypergraph $\mathcal{H} \subseteq [n]^3$ and $\mathbf{P}$, an $(l, t, \gamma, \varepsilon)$-partition of $[n]^2$; recall we write $m = |V_1| = \ldots = |V_t|$. Let P be a triad of $\mathbf{P}$, and let further

$$\mu_P = \frac{|\mathcal{K}_3(P)|}{m^3}. \tag{9}$$

We say that the $(l, t, \gamma, \varepsilon)$-partition $\mathbf{P}$ is (δ, r)-*regular* if

$$\sum \Big\{ \mu_P \colon P \text{ is a } (\delta, r)\text{-irregular triad of } \mathbf{P} \Big\} < \delta \left(\frac{n}{m} \right)^3. \tag{10}$$

For future reference, we make the following remark.

Remark 25. Using the definition of μ_P in (9), the inequality in (10) is easily rewritten as

$$\sum \Big\{ |\mathcal{K}_3(P)| \colon P \text{ is a } (\delta, r)\text{-irregular triad of } \mathbf{P} \Big\} < \delta n^3.$$

We infer from the inequality above that

$$\big| \{ h \in \mathcal{H} \colon h \in \mathcal{K}_3(P) \text{ for some } (\delta, r)\text{-irregular triad } P \text{ of } \mathbf{P} \} \big| < \delta n^3. \tag{11}$$

The following fact appears as Proposition 4.6 in [21] (with a slight but inessential difference in wording) and asserts that in a (δ, r)-regular $(l, t, \gamma, \varepsilon)$-partition there are not too many bipartite graphs P_α^{ij} that fail to be $(l, \varepsilon, 2)$-cylinders, nor are there many triads $P = P_\alpha^{ij} \cup P_\beta^{jk} \cup P_\gamma^{ik}$ that fail to be (δ, r)-regular.

Fact 26. *Let positive constants γ and δ and integers $l > 1/\gamma$ and r be given. There exists $\varepsilon = \varepsilon(l) > 0$ so that, for all integers t, the following holds: whenever $\mathbf{P} = \{P_\alpha^{ij} \colon 0 \le \alpha \le l_{ij} \le l,\ 1 \le i < j \le t\}$ is an $(l, t, \gamma, \varepsilon)$-partition that is (δ, r)-regular with respect to a hypergraph $\mathcal{H}$, we have*

(i)

$$\sum_{1 \le i < j \le t} \big| \{ \alpha \ge 1 \colon P_\alpha^{ij} \text{ is not an } (l, \varepsilon, 2)\text{-cylinder} \} \big| < 3\gamma \binom{t}{2} l,$$

(ii) if I_1 is the set of all triads P of $\mathbf{P}$ that are (δ, r)-irregular with respect to $\mathcal{H}$, then $|I_1| < (2\delta + \gamma) t^3 l^3$.

We update Fact 26 by giving the following easy variation of statement (i).

(i′) Denote by I_2 the set of triads P of $\mathbf{P}$ that contain as a member a bipartite graph P_α^{ij} $(0 \le \alpha \le l_{ij},\ 1 \le i < j \le t)$ satisfying that either $\alpha = 0$ or P_α^{ij} is not an $(l, \varepsilon, 2)$-cylinder. Then if $l > 1/\gamma$ and $\varepsilon = \varepsilon(l)$ is sufficiently small, we have $|I_2| < 2\gamma t^3 l^3$.

Adapting Fact 26 with statement (i') yields the following fact.

Fact 27. *Let $\mathbf{P} = \{P_\alpha^{ij} \colon 0 \le \alpha \le l_{ij} \le l,\ 1 \le i < j \le t\}$ be an $(l, t, \gamma, \varepsilon)$-partition that is (δ, r)-regular with respect to a hypergraph $\mathcal{H}$. Let $I = I_1 \cup I_2$, that is, I is the set of all triads P of $\mathbf{P}$ that are either (δ, r)-irregular with respect to $\mathcal{H}$ or else are such that* *contain as a member a bipartite graph P^{ij}_α $(0 \le \alpha \le l_{ij},\ 1 \le i < j \le t)$ satisfying that either $\alpha = 0$ or P^{ij}_α is not an $(l, \varepsilon, 2)$-cylinder. If $l > 1/\gamma$ and $\varepsilon = \varepsilon(l)$ is sufficiently small, then $|I| < (3\gamma + 2\delta)t^3 l^3$.*

We now state the Regularity Lemma of [11].

Theorem 28. *For every δ and γ with $0 < \gamma \le 2\delta^4$, for all integers t_0 and l_0 and for all integer-valued functions $r = r(t,l)$ and all functions $\varepsilon(l)$, there exist T_0, L_0, and N_0 such that any 3-uniform hypergraph $\mathcal{H} \subseteq [n]^3$, where $n \ge N_0$, admits a $(\delta, r(t,l))$-regular, $(l, t, \gamma, \varepsilon(l))$-partition for some t and l satisfying $t_0 \le t < T_0$ and $l_0 \le l < L_0$.*

For future reference, we state the following two definitions concerning $(l, t, \gamma, \varepsilon)$-regular partitions $\mathbf{P}$ for a triple system $\mathcal{H}$.

Definition 29 ($\Xi = \Xi(\mathbf{P}) = \big\{\xi = \{i,j,k\}_{\alpha\beta\gamma}\big\}$). Let $\mathbf{P} = \{P^{ij}_\alpha : 0 \le \alpha \le l_{ij} \le l,\ 0 \le i < j \le t\}$ be an $(l, t, \gamma, \varepsilon)$-partition of a vertex set V. We write $\Xi = \Xi(\mathbf{P})$ for the set of 6-tuples $\xi = (i, j, k, \alpha, \beta, \gamma)$ with $0 \le \alpha \le l_{ij} \le l$, $0 \le \beta \le l_{jk} \le l$, and $0 \le \gamma \le l_{ik} \le l$, for all $0 \le i < j < k \le t$. Furthermore, we shall write $\{i,j,k\}_{\alpha\beta\gamma}$ for such 6-tuples $\xi \in \Xi$.

Clearly, given an $(l, t, \gamma, \varepsilon)$-partition $\mathbf{P}$ as in Definition 29, we may think of each $\xi \in \Xi = \Xi(\mathbf{P})$ as specifying the triad

$$P_\xi = P^{ij}_\alpha \cup P^{jk}_\beta \cup P^{ik}_\gamma,$$

where $\xi = \{i,j,k\}_{\alpha\beta\gamma}$. Therefore, Ξ may be identified with the family of triads of $\mathbf{P}$.

Definition 30 (Density vector $\vec{s}$). Let $\mathcal{H}$ be a triple system admitting a (δ, r)-regular $(l, t, \gamma, \varepsilon)$-partition $\mathbf{P}$. Let $\vec{s} = (s_\xi)$ be the vector of integers indexed by the $\xi \in \Xi = \Xi(\mathbf{P})$, where $s_\xi = \lfloor \delta^{-1} d_{\mathcal{H}}(P_\xi) \rfloor$ for all $\xi \in \Xi$. We say that $\vec{s}$ is the *density vector* of $\mathcal{H}$ over $\mathbf{P}$.

Let $\vec{s} = (s_\xi)$ be a density vector as above. Then, clearly, if $\xi = \{i,j,k\}_{\alpha\beta\gamma} \in \Xi$, then

$$s_\xi \delta \le d_{\mathcal{H}}(P_\xi) = d_{\mathcal{H}}(P^{ij}_\alpha \cup P^{jk}_\beta \cup P^{ik}_\gamma) < (s_\xi + 1)\delta.$$

3.4. The Counting Lemma

We begin our discussion of the Counting Lemma of [16] by describing the general 'environment' in which this lemma applies. As in [16], we call this environment the Setup.

Setup. For given constants k, δ, l, r, and ε, and a given set $\{\alpha_B : B \in [k]^3\}$ of positive reals, suppose

(*i*) $\mathcal{H}$ is a k-partite 3-cylinder with k-partition $(V_1, \dots, V_k)$ with $|V_1| = \ldots = |V_k| = m$,
(*ii*) $P = \bigcup_{1 \le i < j \le k} P^{ij}$ is an (l, ε, k)-cylinder underlying $\mathcal{H}$,
(*iii*) for all $B \in [k]^3$, the 3-partite 3-cylinder $\mathcal{H}(B)$ is (α_B, δ, r)-regular with respect to the triad $P(B)$ (see Definition 21).

The main result of [16] is the following Counting Lemma.

Lemma 31 (The Counting Lemma). *Let $k \geq 4$ be a fixed integer. For all constants α and $\beta > 0$, there exists $\delta > 0$ such that, for all integers $l \geq 1/\delta$, there exist r and $\varepsilon > 0$ so that, whenever a k-partite 3-cylinder $\mathcal{H}$ with an underlying cylinder P satisfies the conditions of the* Setup *with constants k, δ, l, r, and ε, and set $\{\alpha_B\colon B \in [k]^3\}$, where $\alpha_B \geq \alpha$ for all $B \in [k]^3$, we have*

$$(1-\beta)l^{-\binom{k}{2}}m^k \prod_{B\in[k]^3} \alpha_B \leq |\mathcal{K}_k(\mathcal{H})| \leq (1+\beta)l^{-\binom{k}{2}}m^k \prod_{B\in[k]^3} \alpha_B. \tag{12}$$

For our purposes in this paper, we are content to know that under the hypotheses of the Counting Lemma, at least one copy of $K_k^{(3)}$ is ensured within $\mathcal{H}$, provided m is sufficiently large.

4. Proof of Theorem 1

Recall that in Theorem 1, we promised to show that for any hereditary property Π of triple systems,

$$\log_2 |\Pi_n| = \mathrm{ex}_{\mathrm{ind}}(n, \mathbb{F}) + o(n^3), \tag{13}$$

where $\mathbb{F}$ is the forbidden class of triple systems associated with Π. To that end, let Π be a fixed hereditary property of triple systems, and let $\mathbb{F}$ be its class of forbidden triple systems. We show the equality in (13). We begin by first showing the easy lower bound

$$\log_2 |\Pi_n| \geq \mathrm{ex}_{\mathrm{ind}}(n, \mathbb{F}). \tag{14}$$

Let $[n]^3 = \mathcal{G} \cup \mathcal{M} \cup \mathcal{N}$ be a partition of $[n]^3$ that gives the value of $\mathrm{ex}_{\mathrm{ind}}(n, \mathbb{F})$. More explicitly, let $\mathcal{G} \subseteq [n]^3$ be of size $|\mathcal{G}| = \mathrm{ex}_{\mathrm{ind}}(n, \mathbb{F})$, with the property that there exist disjoint sets $\mathcal{M}$ and $\mathcal{N} \subseteq [n]^3 \setminus \mathcal{G}$, such that $\mathcal{M} \cup \mathcal{N} = [n]^3 \setminus \mathcal{G}$, so that, for all $\mathcal{G}' \subseteq \mathcal{G}$, no $\mathcal{F} \in \mathbb{F}$ is an induced subhypergraph of $\mathcal{M} \cup \mathcal{G}'$. Since there are $2^{|\mathcal{G}|} = 2^{\mathrm{ex}_{\mathrm{ind}}(n,\mathbb{F})}$ subhypergraphs $\mathcal{G}' \subseteq \mathcal{G}$, and each such $\mathcal{G}'$ is such that $\mathcal{G}' \cup \mathcal{M} \in \Pi_n$, the inequality in (14) is established. (Clearly, this simple argument holds for all s, and not simply for $s = 3$.)

What remains to be shown is the more difficult upper bound. We show that, for any $\nu > 0$, the inequality

$$\log_2 |\Pi_n| \leq \mathrm{ex}_{\mathrm{ind}}(n, \mathbb{F}) + \nu n^3 \tag{15}$$

holds for sufficiently large n.

To that end, let $\nu > 0$ be given. To show (15), we first define some auxiliary parameters. We define positive constants σ, μ, α_0, θ, τ, δ, and γ, and positive integers k_0, l_0, t_0, as well as functions $\varepsilon = \varepsilon(l)$ and $r = r(l)$. The following description of these auxiliary parameters is rather technical. The reader may find it useful to have in mind that these constants obey the 'hierarchy' given in (16) and (17) below:

$$\nu \gg \sigma,\, \mu \gg \frac{1}{k_0},\, \alpha_0,\, \tau,\, \theta \gg \delta > \gamma > \frac{1}{l_0} \gg \frac{1}{r} \gg \varepsilon, \tag{16}$$

$$\gamma > \frac{1}{t_0}. \tag{17}$$

In what follows, we shall be considering hypergraphs with n vertices, where n is such that

$$\varepsilon, \ \frac{1}{t_0} \ggg \frac{1}{n}.$$

4.1. Definition of the Auxiliary Constants

For the given constant $\nu > 0$, define positive constants

$$\begin{aligned} \mu &= \mu(\nu), & (18)\\ \sigma &= \sigma(\nu), & (19)\\ \theta_1 &= \theta_1(\nu), & (20)\\ \tau &= \tau(\nu), & (21) \end{aligned}$$

all smaller than 1, such that

$$2\theta_1 + 9\mu + 4\sigma \log \frac{2e}{\sigma} < \frac{\nu}{2}, \tag{22}$$

and

$$\tau \le \mu^3 < \sigma < \frac{1}{8}. \tag{23}$$

With μ, σ, θ_1, and τ as in (22) and (23), the following inequality holds for all sufficiently large integers n:

$$\begin{aligned} \big(\mathrm{ex_{ind}}(n, \mathbb{F}) + 4\mu n^3 + \tau n^3\big)(1 + \theta_1) + n^2 \log n + 4n^3 \sigma \log \frac{2e}{\sigma} \\ < \mathrm{ex_{ind}}(n, \mathbb{F}) + \nu n^3. \qquad (24) \end{aligned}$$

For the constant $\tau > 0$ in (21), let

$$k_0 = k_0(\tau) \tag{25}$$

be an integer satisfying that for all integers $n \ge t \ge k_0$, we have

$$\left| \binom{t}{3}^{-1} \mathrm{ex_{ind}}(t, \mathbb{F}) - \binom{n}{3}^{-1} \mathrm{ex_{ind}}(n, \mathbb{F}) \right| < \tau. \tag{26}$$

Note that the existence of k_0 follows from the fact that $\mathrm{ex_{ind}}(\mathbb{F}) = \lim_{n\to\infty} \binom{n}{3}^{-1} \mathrm{ex_{ind}}(n, \mathbb{F})$ exists for any fixed family of triple systems $\mathbb{F}$.

For the constants μ and σ given in (18) and (19), let positive constants θ_2, δ_1, and α_0, and a positive integer t_1 satisfy

$$\begin{aligned} \frac{2}{t_1} &< \delta_1, & (27)\\ 2\delta_1^{1/2} \binom{k_0}{3} &\le \mu^3, & (28)\\ \frac{\alpha_0}{3} &\le \frac{\sigma}{2} & (29)\\ 4\delta_1 &< \frac{\sigma}{2}, & (30) \end{aligned}$$

$$(31)$$

and

$$(2\theta_2 + \alpha_0 - \theta_2\alpha_0) \log \frac{2e}{2\theta_2 + \alpha_0 - \theta_2\alpha_0} \leq \sigma \log \frac{2e}{\sigma}, \tag{32}$$

and, finally,

$$100\delta_1 + \frac{100\delta_1}{\mu^2} < \delta_1^{1/2}. \tag{33}$$

Set

$$\begin{aligned} t_0' &= \max\{k_0, t_1\}, && (34)\\ \theta &= \min\{\theta_1, \theta_2\}. && (35) \end{aligned}$$

Some of the remainder of our discussion on defining the auxiliary constants will concern the Counting Lemma, Lemma 31. Recall that regarding the constants involved, Lemma 31 states

$$\text{“}\forall k\ \forall \alpha, \beta\ \exists \delta \colon \forall l > \frac{1}{\delta}\ \exists\, r, \varepsilon \text{ so that } \ldots \text{”}.$$

Set

$$k = k_0 \tag{36}$$

where k_0 is defined in (25). For an integer $f_0 \leq k_0$, the constant α_0 given in (30) and (32), and $\beta = 1/2$, let

$$\delta_{2,f_0} = \delta_{31}(f_0, \alpha_0)$$

be that constant guaranteed by Lemma 31. Set

$$\delta_2 = \min\{\delta_{2,f_0} \colon 4 \leq f_0 \leq k_0\}. \tag{37}$$

Set

$$\delta = \min\{\delta_1, \delta_2\}, \tag{38}$$

where δ_1 and δ_2 were given in (27)–(33) and (37). We now let

$$t_0 = \max\{t_0', \lceil 2/\delta \rceil\}. \tag{39}$$

Set

$$\gamma = \frac{\delta}{3} \tag{40}$$

and let

$$l_0 > \frac{1}{\gamma} > \frac{1}{\delta} \tag{41}$$

be any integer. We now define functions $r(l)$ and $\varepsilon(l)$.

For an integer $4 \leq f_0 \leq k_0$ and any integer $l \geq l_0 > 1/\gamma > 1/\delta$, Lemma 31 gives us

$$\begin{aligned} r_{f_0}(l) &= r_{31}(f_0, \alpha_0, \delta, l),\\ \varepsilon_{f_0}'(l) &= \varepsilon_{31}(f_0, \alpha_0, \delta, l). \end{aligned}$$

For any integer $l \geq l_0 > 1/\gamma > 1/\delta$, set

$$r(l) = \max\{r_{f_0}(l) \colon 4 \leq f_0 \leq k_0\}, \tag{42}$$

$$\varepsilon'(l) \quad = \quad \min\{\varepsilon'_{f_0}(l) \colon 4 \leq f_0 \leq k_0\}. \tag{43}$$

For the definition of $\varepsilon(l)$, recall Fact 15, which guarantees to every k, l, and θ, an $\varepsilon > 0$ such that, for any (l, ε, k)-cylinder P, relation (6) holds. For $k = 3$ and θ given in (35), let, for any integer l,

$$\varepsilon_{15} = \varepsilon_{15}(l) \tag{44}$$

be that function whose existence is guaranteed by Fact 15. For $l > 1/\gamma$, let

$$\varepsilon_{27} = \varepsilon_{27}(l) \tag{45}$$

be that function guaranteed by Fact 27. Set

$$\varepsilon = \varepsilon(l) = \min\{\varepsilon'(l), \varepsilon_{15}(l), \varepsilon_{27}(l)\}. \tag{46}$$

This concludes the definitions of the constants σ, μ, k_0, α_0, θ, δ, l_0, γ, and t_0, and functions $r = r(l)$ and $\varepsilon = \varepsilon(l)$. We note that, with the constants described above, we may assume that the hierarchy in (16) and (17) is satisfied.

4.2. Proof of (15)

We begin our proof of the upper bound

$$\log_2 |\Pi_n| \leq \mathrm{ex}_{\mathrm{ind}}(n, \mathbb{F}) + \nu n^3$$

as follows. For each $\mathcal{H} \in \Pi_n$, we use Theorem 28 to obtain a (δ, r)-regular $(l, t, \gamma, \varepsilon)$-partition $\mathbf{P} = \mathbf{P}_{\mathcal{H}}$. To that end, we must first disclose the input constants and functions required by Theorem 28. Recall that regarding the constants involved, Theorem 28 states that

$$\text{“}\forall \delta, \gamma, t_0, l_0, r(t,l), \varepsilon(l)\ \exists T_0, L_0, N_0 \text{ such that } \ldots \text{”}.$$

Let δ be the constant given in (38), γ be the constant given in (40), t_0 be the integer given in (39), and l_0 be the integer given in (41). Furthermore, let $r = r(l)$ be the function given in (42) and $\varepsilon = \varepsilon(l)$ be the function given in (46).

With the above disclosed input values, Theorem 28 guarantees constants T_0, L_0, and N_0 so that any triple system $\mathcal{H}$ on $n > N_0$ vertices admits a (δ, r)-regular, $(l, t, \gamma, \varepsilon)$-partition $\mathbf{P}$, for some integers t and l satisfying $t_0 \leq t \leq T_0$ and $l_0 \leq l \leq L_0$. We stress that what is important for us is that, provided $n > N_0$, *every* $\mathcal{H} \in \Pi_n$ admits a (δ, r)-regular, $(l, t, \gamma, \varepsilon)$-partition $\mathbf{P} = \mathbf{P}(\mathcal{H})$. Whenever $\mathcal{H} \in \Pi_n$ admits many such partitions $\mathbf{P}$, we arbitrarily choose one partition $\mathbf{P} = \mathbf{P}(\mathcal{H})$ to associate with $\mathcal{H}$. It will be these partitions that will allow us to estimate $|\Pi_n|$ effectively.

Our strategy for estimating $|\Pi_n|$ is as follows. For $n > N_0$, where N_0 was determined by Theorem 28, and for the family Π_n, let $\check{\mathbf{P}} = \{\mathbf{P}_1, \ldots, \mathbf{P}_p\}$ be the set of all (δ, r)-regular, $(l, t, \gamma, \varepsilon)$-partitions guaranteed by Theorem 28 for the hypergraphs $\mathcal{H} \in \Pi_n$. The crucial observation is that $\check{\mathbf{P}}$ induces a partition of Π_n into equivalence classes

$$\{C(i, \vec{s}) \colon 1 \leq i \leq p \text{ and } \vec{s} \text{ a density vector for } \mathbf{P}_i\},$$

defined in the following way: for each $i \in [p]$ and density vector $\vec{s}$, for all $\mathcal{H} \in \Pi_n$,

$$\mathcal{H} \in C(i, \vec{s}) \Longleftrightarrow \begin{cases} \mathbf{P}_i \in \check{\mathbf{P}} \text{ is the partition } \mathbf{P}(\mathcal{H}) \text{ associated with } \mathcal{H}, \\ \text{and } \mathcal{H} \text{ has density vector } \vec{s} \text{ over } \mathbf{P}_i. \end{cases}$$

We note that here we see why it was important to choose a unique partition to associate with each $\mathcal{H} \in \varPi_n$. For convenience, denote by $\vec{S}$ the set of all possible density vectors (i.e., all vectors of length $\binom{t}{3}l^3$ whose coordinates take the integer values between 0 and $\lfloor 1/\delta \rfloor$).

In view of the partition of $\varPi_n$ into the equivalence classes $\{C(i,\vec{s})\colon 1 \le i \le p,\ \vec{s} \in \vec{S}\}$, we divide the process of estimating $|\varPi_n|$ into two tasks: we estimate the number of such equivalence classes $q = \left|\{C(i,\vec{s})\colon 1 \le i \le p,\ \vec{s} \in \vec{S}\}\right|$, and we give a uniform upper bound for the cardinality of all these classes $C(i,\vec{s})$ $(1 \le i \le p,\ \vec{s} \in \vec{S})$. For the first task, we show that

$$\log_2 q = O(n^2), \tag{47}$$

and, for the second task, we show that

$$\log_2 |C(i,\vec{s})| = \big(\mathrm{ex}_{\mathrm{ind}}(n,\mathbb{F}) + \tau n^3 + 4\mu n^3\big)(1+\theta) + 4n^3\sigma \log \frac{2e}{\sigma} + O(1), \tag{48}$$

for any $C(i,\vec{s})$ $(1 \le i \le p,\ \vec{s} \in \vec{S})$. Therefore, as a result of (47) and (48), combined with the inequality in (24), we have

$$\begin{aligned}
\log_2 |\varPi_n| &= \log_2 \sum_{i=1}^{p} \sum_{\vec{s} \in \vec{S}} |C(i,\vec{s})|, \\
&\le O(n^2) + \big(\mathrm{ex}_{\mathrm{ind}}(n,\mathbb{F}) + \tau n^3 + 4\mu n^3\big)(1+\theta) + 4n^3\sigma \log \frac{2e}{\sigma}, \\
&\le \big(\mathrm{ex}_{\mathrm{ind}}(n,\mathbb{F}) + \tau n^3 + 4\mu n^3\big)(1+\theta) + n^2 \log n + 4n^3\sigma \log \frac{2e}{\sigma}, \\
&\le \mathrm{ex}_{\mathrm{ind}}(n,\mathbb{F}) + \nu n^3.
\end{aligned} \tag{49}$$

Thus, we are finished proving (15) once we have established (47) and (48). We do this below, and begin by estimating the value q.

4.2.1. Estimation of q. The number of partitions of the vertex set that arise from regular partitions $\mathbf{P} \in \check{\mathbf{P}}$ is at most $(T_0+1)^n$. Moreover, Theorem 28 guarantees families of graphs P_α^{ij} $(0 \le \alpha \le l_{ij} \le L_0,\ 0 \le i < j \le t \le T_0)$ with at most $\binom{T_0+1}{2}(L_0+1)$ members. Since there are at most $\delta^{-\binom{T_0+1}{3}(L_0+1)^3}$ density vectors that could be associated to any one $\mathbf{P} \in \{\mathbf{P}_1, \ldots, \mathbf{P}_p\}$, the number q of classes is at most

$$\left(\frac{1}{\delta}\right)^{\binom{T_0+1}{3}(L_0+1)^3} (T_0+1)^n \left(\binom{T_0+1}{2}(L_0+1)\right)^{\binom{n}{2}} = 2^{O(n^2)}.$$

Thus, the bound

$$\log_2 q = O(n^2).$$

is established.

4.2.2. Estimation of $|C|$. Our second task is to estimate $|C(i,\vec{s})|$ for all $C(i,\vec{s})$ $(1 \le i \le p,\ \vec{s} \in \vec{S})$. To that effect, fix one such $C(i,\vec{s})$ and, for simplicity of notation, set $C = C(i,\vec{s})$ and $\mathbf{P} = \mathbf{P}_i$. Let $\mathbf{P}$ have equitable partition $[n] = V_0 \cup \ldots \cup V_t$, where, as usual, $m = |V_1| = \ldots = |V_t| = \lfloor n/t \rfloor$, and let the system of bipartite graphs of $\mathbf{P}$ be P_α^{ij} $(1 \le i < j \le t,\ 0 \le \alpha \le l_{ij} \le l)$.

We shall estimate from above the number of $\mathcal{H} \in \Pi_n$ that are members of the equivalence class C. The main tool we use to bound $|C|$ is that the partition $\mathbf{P}$ is a (δ, r)-regular, $(l, t, \gamma, \varepsilon)$-partition with respect to *all* $\mathcal{H} \in C$ and that all $\mathcal{H} \in C$ have density vector $\vec{s}$ over $\mathbf{P}$. Owing to this highly regular structure of $\mathbf{P}$, we are able to give an upper bound for $|C|$ in an effective way. We shall show that

$$\log_2 |C| \leq \big(\mathrm{ex}_{\mathrm{ind}}(n, \mathbb{F}) + \tau n^3 + 4\mu n^3\big)(1+\theta) + 4n^3\sigma \log \frac{2e}{\sigma} + O(1). \tag{50}$$

Before we plunge into the proof of (50), we give a quick overview of our strategy. Recall we write $\Xi = \Xi(\mathbf{P}) = \{\xi\}$ for the system of $\xi = \{i, j, k\}_{\alpha\beta\gamma}$, with each such ξ specifying a triad of $\mathbf{P}$. For any fixed $\mathcal{H} \in C$, we shall show that we may associate a partition

$$\Xi = \Xi(\mathbf{P}) = I \cup G \cup N \cup M \tag{51}$$

of Ξ with the properties we describe below. For simplicity, for any $\Xi' \subset \Xi$, we write $\mathcal{K}_3(\Xi')$ for

$$\bigcup_{\xi \in \Xi'} \mathcal{K}_3(P_\xi),$$

where, we recall, P_ξ is the triad $P_\alpha^{ij} \cup P_\beta^{jk} \cup P_\gamma^{ik}$ if $\xi = \{i, j, k\}_{\alpha\beta\gamma}$.

(P_1) The system $\mathcal{I} = \mathcal{H} \cap \mathcal{K}_3(I)$ is such that $|\mathcal{I}| \leq 3\delta n^3$.

(P_2) The system $\mathcal{N} = \mathcal{H} \cap \mathcal{K}_3(N)$ is such that $|\mathcal{N}| \leq (\alpha_0/3)n^3$.

(P_3) The system $\mathcal{M} = \mathcal{H} \cap \mathcal{K}_3(M)$ is such that

$$\big|\mathcal{K}_3(M) \setminus \mathcal{M}\big| \leq (2\theta + \alpha_0 - \alpha_0\theta)\big|\mathcal{K}_3(M)\big|. \tag{52}$$

(P_4) The system $G = \Xi \setminus (I \cup N \cup M)$ is such that (59) below holds. Let $\mathcal{G} = \mathcal{H} \cap \mathcal{K}_3(G)$.

(P_5) Let

$$\mathcal{E}_0 = [n]^3 \setminus \big(\mathcal{G} \cup \mathcal{I} \cup \mathcal{N} \cup \mathcal{M}\big).$$

Then $|\mathcal{E}_0| \leq \delta n^3$.

A little thought now shows that the existence of a partition as in (51), satisfying properties (P_1)–(P_5), for every $\mathcal{H} \in C$, may be used to bound $|C|$ from above. Indeed, note first that we have established a partition

$$\mathcal{H} = \mathcal{E}_0 \cup \mathcal{I} \cup \mathcal{N} \cup \mathcal{M} \cup \mathcal{G}$$

for any given $\mathcal{H} \in C$. Observe also that (i) the number of partitions of Ξ into 4 parts as in (51) is at most $4^{|\Xi|} = O(1)$, (ii) the number of choices we have for the triples in $\mathcal{I} \cup \mathcal{N} \cup \mathcal{M}$ is 2^{cn^3}, where $c = c(\delta, \alpha_0, \theta) \to 0$ as δ, α_0, $\theta \to 0$, (iii) once G is fixed, the number of choices we have for the triples in $\mathcal{G}$ is

$$\leq 2^{|\mathcal{K}_3(G)|}, \tag{53}$$

which may be estimated using (59) and the fact that all $\xi \in G$ are such that the associated triad P_ξ is an $(l, \varepsilon, 3)$-cylinder, and (iv) the number of choices we have for the triples in $\mathcal{E}_0$ is at most $2^{\delta n^3}$. To obtain an upper bound for $|C|$ it then suffices to put together observations (i)–(iv). We do this in detail in what follows.

Fix $\mathcal{H} \in C$.

Definition of I. We put $\xi = \{i,j,k\}_{\alpha\beta\gamma} \in \Xi$ in I if any of the following conditions holds:

(*i*) $\alpha = 0$, $\beta = 0$, or $\gamma = 0$,
(*ii*) P^{ij}_α, P^{jk}_β, or P^{ik}_γ is not an $(l,\varepsilon,2)$-cylinder,
(*iii*) $P_\xi = P^{ij}_\alpha \cup P^{jk}_\beta \cup P^{ik}_\gamma$ is not a (δ, r)-regular triad with respect to $\mathcal{H}(\{i,j,k\})$.

Note that it follows from Fact 27 that since $l \geq l_0 > 1/\gamma$ (see (41)), $\varepsilon \leq \varepsilon_{27}$ (see (46)), and $\gamma = \delta/3$ (see (40)) that

$$|I| < (3\gamma + 2\delta)t^3 l^3 < 3\delta t^3 l^3.$$

Note that, because $t \geq 5$, we may further conclude that

$$|I| < 100\delta l^3 \binom{t}{3}. \tag{54}$$

As it turns out, we shall also be interested in viewing $I \subset \Xi$ as a hypergraph on $[t] = \{1,\dots,t\}$, where we allow multiple, or parallel, edges: for each $\{i,j,k\}$ $(1 \leq i < j < k \leq t)$, as many as l^3 parallel triples

$$\{i,j,k\}_{\alpha\beta\gamma} \in I$$

are possible.

We now put $\mathcal{I} = \mathcal{H} \cap \mathcal{K}_3(I)$. It follows from statements (*iii*) and (*iv*) of Definition 23 and (11) from Remark 25 that

$$|\mathcal{I}| \leq 4\gamma \binom{t}{2} \left(\frac{n}{t}\right)^2 n + \delta n^3 \leq 3\delta n^3. \tag{55}$$

Definition of N. We put $\xi = \{i,j,k\}_{\alpha\beta\gamma} \in \Xi \setminus I$ in N if P_ξ is a 'thin' regular triad, that is, a regular triad that captures few triples of $\mathcal{H}$. More precisely, we put $\xi = \{i,j,k\}_{\alpha\beta\gamma} \in \Xi \setminus I$ in N if

$$d_{\mathcal{H}}(P_\xi) < \alpha_0,$$

where α_0 is given in (29) and (32).

We now put $\mathcal{N} = \mathcal{H} \cap \mathcal{K}_3(N)$. Note that by the definition of N, every triad P_ξ with $\xi \in N$ is an $(l,\varepsilon,3)$-cylinder. It follows from Fact 15 that for the constant θ in (35) and our choice of ε in (46), we have

$$\frac{m^3}{l^3}(1-\theta) < |\mathcal{K}_3(P_\xi)| < \frac{m^3}{l^3}(1+\theta).$$

Therefore, we conclude that $|\mathcal{N}|$ satisfies

$$|\mathcal{N}| \leq \alpha_0 \binom{t}{3} l^3 \left(\frac{n}{t}\right)^3 \frac{1}{l^3}(1+\theta) \leq \frac{\alpha_0}{3} n^3. \tag{56}$$

As for I, we shall be considering N as a multi-hypergraph on the vertex set $[t]$ (that is, a multi-set on $[t]^3$).

Definition of M. We put $\xi = \{i,j,k\}_{\alpha\beta\gamma} \in \Xi \setminus I$ in M if P_ξ is a 'thick' regular triad, that is, a regular triad that captures many triples of $\mathcal{H}$. More precisely, we put $\xi =$

$\{i,j,k\}_{\alpha\beta\gamma} \in \Xi \setminus I$ in M if

$$d_{\mathcal{H}}(P_\xi) > 1 - \alpha_0,$$

where α_0 is given in (29) and (32). We also put $\mathcal{M} = \mathcal{H} \cap \mathcal{K}_3(M)$.

We are interested in estimating $|\mathcal{M}|$. Set

$$c_{\mathcal{M}} = |M| \bigg/ \binom{t}{3} l^3 .$$

As when dealing with N and $\mathcal{N}$, we observe that every triad P_ξ with $\xi \in M$ is an $(l, \varepsilon, 3)$-cylinder. Applying Fact 15 to each such triad P_ξ we conclude that

$$\frac{m^3}{l^3}(1-\theta) < |\mathcal{K}_3(P_\xi)| < \frac{m^3}{l^3}(1+\theta),$$

and hence

$$|M|\frac{m^3}{l^3}(1-\theta) < |\mathcal{K}_3(M)| < |M|\frac{m^3}{l^3}(1+\theta). \tag{57}$$

Consequently,

$$c_{\mathcal{M}}\binom{t}{3} l^3 (1-\alpha_0)\frac{m^3}{l^3}(1-\theta) \le |\mathcal{M}| \le c_{\mathcal{M}}\binom{t}{3} l^3 \frac{m^3}{l^3}(1+\theta). \tag{58}$$

Finally, M may also be considered as a multi-hypergraph on the vertex set $[t]$.

Definition of G. We now let $G = \Xi \setminus (I \cup N \cup M)$. Thus, $\xi \in G$ if P_ξ is a regular triad with density with respect to $\mathcal{H}$ bounded away from 0 and 1 by at least α_0. More precisely, we have $\xi = \{i,j,k\}_{\alpha\beta\gamma} \in G$ if and only if

(*i*) $\alpha, \beta, \gamma > 0$ and $P_\xi = P_\alpha^{ij} \cup P_\beta^{jk} \cup P_\gamma^{ik}$ is an $(l, \varepsilon, 3)$-cylinder,

(*ii*) $\mathcal{H}$ is $(\bar{\alpha}, \delta, r)$-regular with respect to P_ξ, where $\bar{\alpha} = \bar{\alpha}(\xi)$ satisfies $\alpha_0 \le \bar{\alpha} \le 1 - \alpha_0$.

We now state the following claim about G.

Lemma 32. *With $\mu > 0$ as defined in* (18), *we have*

$$|G| \le \left(\mathrm{ex}_{\mathrm{ind}}(t, \mathbb{F}) + 4\mu\binom{t}{3} \right) l^3. \tag{59}$$

Owing to the technical nature of our proof of Lemma 32 above, we give its proof in a separate section (see Section 4.2.3) and continue proving (50). Before we proceed, let us remark that, as with I, N and M, the system G may be viewed as a multi-hypergraph on $[t]$.

We now put $\mathcal{G} = \mathcal{H} \cap \mathcal{K}_3(G)$. The main use of Lemma 32 is in estimating the total number of triple systems $\mathcal{G}$ that may arise in this form, *once G is fixed.*

To that end, consider

$$\mathcal{K}_3(G) = \bigcup_{\xi \in G} \mathcal{K}_3(P_\xi).$$

It follows from Fact 15 that, for the constant θ in (35) and our choice of ε in (46), we have

$$|\mathcal{K}_3(P_\xi)| < \frac{m^3}{l^3}(1+\theta). \tag{60}$$

Using Lemma 32 and (60), we see that

$$\begin{aligned}|\mathcal{K}_3(G)| &\leq \frac{m^3}{l^3}(1+\theta)|G|,\\ &\leq m^3\left(\mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F})+4\mu\binom{t}{3}\right)(1+\theta).\end{aligned}$$

The easy but essential observation now is that, if G is fixed, the total number of systems $\mathcal{G}$ that may arise as $\mathcal{H}\cap\mathcal{K}_3(G)$, as we vary $\mathcal{H}\in C$, is at most

$$\begin{aligned}2^{|\mathcal{K}_3(G)|} &\leq 2^{m^3\left(\mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F})+4\mu\binom{t}{3}\right)(1+\theta)}\\ &\leq 2^{(n/t)^3\left(\mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F})+4\mu\binom{t}{3}\right)(1+\theta)} \qquad (61)\end{aligned}$$

(see (53)).

Definition of $\mathcal{E}_0$. We define $\mathcal{E}_0$ to consist of all triples $e\in\mathcal{H}$ that do not occur in

$$\mathcal{K}_3(\Xi)=\bigcup_{\xi\in\Xi}\mathcal{K}_3(P_\xi). \qquad (62)$$

More explicitly, a triple $e\in\mathcal{H}$ is in $\mathcal{E}_0$ if either

(i) $e\cap V_0\neq\emptyset$, or else

(ii) $|e\cap V_i|\geq 2$ for some $i\in[t]$.

Note that $|\mathcal{E}_0|$ satisfies

$$|\mathcal{E}_0|\leq T_0n^2+nt\left(\frac{n}{t}\right)^2\leq\left(o(1)+\frac{1}{t_0}\right)n^3\leq\delta n^3, \qquad (63)$$

where the last inequality follows from (39).

Final estimate for $|C|$. We now put together the information we have to prove (50). We start by recalling that the number of partitions of $\Xi=\Xi(\mathbf{P})$ is

$$4^{|\Xi|}=O(1). \qquad (64)$$

The number of choices that there exist for $\mathcal{E}_0=\mathcal{E}_0(\mathcal{H})$ $(\mathcal{H}\in C)$ is, because of (63),

$$\leq\binom{\binom{n}{3}}{\delta n^3}. \qquad (65)$$

The number of choices that there exist for $\mathcal{I}=\mathcal{I}(\mathcal{H})$ $(\mathcal{H}\in C)$ is, because of (55),

$$\leq\binom{\binom{n}{3}}{3\delta n^3}. \qquad (66)$$

The number of choices that there exist for $\mathcal{N}=\mathcal{N}(\mathcal{H})$ $(\mathcal{H}\in C)$ is, because of (56),

$$\leq\binom{\binom{n}{3}}{(\alpha_0/3)n^3}. \qquad (67)$$

Additionally, recalling (52), (58), and (57), we see that the number of choices that there exist for $\mathcal{M}=\mathcal{M}(\mathcal{H})$ $(\mathcal{H}\in C)$, once $M\subset\Xi$ is fixed, is

$$\leq\binom{|\mathcal{K}_3(M)|}{(2\theta+\alpha_0-\alpha_0\theta)\big|\mathcal{K}_3(M)\big|}$$

$$\begin{aligned}
&\le \binom{c_{\mathcal{M}(\mathcal{H})}\binom{t}{3}m^3(1+\theta)}{c_{\mathcal{M}(\mathcal{H})}\binom{t}{3}m^3(1-\theta)(1-\alpha_0)} \\
&= \binom{c_{\mathcal{M}(\mathcal{H})}\binom{t}{3}m^3(1+\theta)}{c_{\mathcal{M}(\mathcal{H})}\binom{t}{3}m^3\big(1+\theta-(1-\theta)(1-\alpha_0)\big)} \\
&\le \left(\frac{e(1+\theta)}{2\theta+\alpha_0-\theta\alpha_0}\right)^{(2\theta+\alpha_0-\theta\alpha_0)n^3} \\
&\le 2^{(2\theta+\alpha_0-\theta\alpha_0)n^3\log(2e/(2\theta+\alpha_0-\theta\alpha_0))}. \qquad (68)
\end{aligned}$$

Because of the inequality in (32), we further conclude that the quantity in (68) is

$$\le 2^{\sigma n^3\log(2e/\sigma)} < 2^{2\sigma n^3\log(2e/\sigma)}. \qquad (69)$$

Combining (61) with (64)–(67), and (69), we infer that

$$\log_2|C| \le \left(\mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F}) + 4\mu\binom{t}{3}\right)\frac{n^3}{t^3}(1+\theta) + 4n^3\sigma\log\frac{2e}{\sigma} + O(1). \qquad (70)$$

The major work in bounding $|C|$ from above is finished. However, to establish (50), we need to replace $\mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F})$ by $\mathrm{ex}_{\mathrm{ind}}(n,\mathbb{F})$. Recall that we chose k_0 in (25) so that, for the auxiliary constant $\tau > 0$ in (21), we have

$$\left|\binom{t}{3}^{-1}\mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F}) - \binom{n}{3}^{-1}\mathrm{ex}_{\mathrm{ind}}(n,\mathbb{F})\right| < \tau$$

for all $n \ge t \ge k_0$ or, equivalently,

$$\left|\binom{t}{3}^{-1}\binom{n}{3}\mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F}) - \mathrm{ex}_{\mathrm{ind}}(n,\mathbb{F})\right| < \tau\binom{n}{3}.$$

With n sufficiently large, the above inequality implies

$$\mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F})\frac{n^3}{t^3} \le \mathrm{ex}_{\mathrm{ind}}(n,\mathbb{F}) + \tau n^3. \qquad (71)$$

Combining (70) with (71) we obtain

$$\log_2|C| \le (\mathrm{ex}_{\mathrm{ind}}(n,\mathbb{F}) + \tau n^3 + 4\mu n^3)(1+\theta) + 4n^3\sigma\log\frac{2e}{\sigma} + O(1).$$

Therefore, inequality (50) follows.

We now return to the task of proving Lemma 32.

4.2.3. Proof of Lemma 32. The proof of Lemma 32 will require some work. We shall in fact make use of two auxiliary propositions (Propositions 33 and 36) and two auxiliary claims (Claims 34 and 35). The proofs of these results are given in later sections.

Let us now start the proof of Lemma 32. We shall suppose for a contradiction that we in fact have

$$|G| > \left(\mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F}) + 4\mu\binom{t}{3}\right)l^3. \qquad (72)$$

Recall we have the partition $\Xi = I \cup G \cup N \cup M$ of $\Xi = \Xi(\mathbf{P})$ as in (51). Moreover, we recall that we may think of I, G, N, and M as multi-hypergraphs.

We prove the proposition below, which extends Fact 4.1 of [15]. Let ψ: $[t]^2 \to [l]$ be any 'choice' function. We say that $\xi = \{i,j,k\}_{\alpha\beta\gamma} \in \Xi$ is *chosen* by ψ if

$$\psi(\{i,j\}) = \alpha, \qquad \psi(\{j,k\}) = \beta, \quad \text{and} \quad \psi(\{i,k\}) = \gamma.$$

Proposition 33. *There exists* ψ: $[t]^2 \to [l]$ *such that*

$$\big|\{\xi \in G \colon \xi \text{ is chosen by } \psi\}\big| \geq \frac{|G|}{l^3}\left(1 - \frac{\mu}{2}\right),$$

and

$$\big|\{\xi \in I \colon \xi \text{ is chosen by } \psi\}\big| \leq \delta^{1/2}\binom{t}{3}.$$

We shall prove Proposition 33 in Section 4.2.4. Let us continue with the proof of Lemma 32. Recall our (contrary) assumption in (72). Let ψ: $[t]^2 \to [l]$ be the function guaranteed by Proposition 33, and define G^ψ, I^ψ, N^ψ, and M^ψ as

$$\begin{aligned} G^\psi &= \big\{\{i,j,k\} \in [t]^3 \colon \exists \xi = \{i,j,k\}_{\alpha\beta\gamma} \in G \text{ such that } \psi \text{ chooses } \xi\big\} \\ I^\psi &= \big\{\{i,j,k\} \in [t]^3 \colon \exists \xi = \{i,j,k\}_{\alpha\beta\gamma} \in I \text{ such that } \psi \text{ chooses } \xi\big\} \\ N^\psi &= \big\{\{i,j,k\} \in [t]^3 \colon \exists \xi = \{i,j,k\}_{\alpha\beta\gamma} \in N \text{ such that } \psi \text{ chooses } \xi\big\} \\ M^\psi &= \big\{\{i,j,k\} \in [t]^3 \colon \exists \xi = \{i,j,k\}_{\alpha\beta\gamma} \in M \text{ such that } \psi \text{ chooses } \xi\big\} \end{aligned} \tag{73}$$

Note that G^ψ, I^ψ, N^ψ and M^ψ are each triple systems on the vertex set $[t] = \{1,\dots,t\}$, without parallel triples. Note that $[t]^3 = G^\psi \cup I^\psi \cup N^\psi \cup M^\psi$ is a partition. We infer from (72) and Proposition 33 that

$$\begin{aligned} |G^\psi| &\geq \frac{|G|}{l^3}\left(1 - \frac{\mu}{2}\right) \\ &> \left(\mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F}) + 4\mu\binom{t}{3}\right)\left(1 - \frac{\mu}{2}\right), \\ &\geq \mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F}) + 2\mu\binom{t}{3}, \end{aligned} \tag{74}$$

and

$$|I^\psi| \leq \delta^{1/2}\binom{t}{3}.$$

If we could ensure $I^\psi = \emptyset$ now, we could immediately begin working on producing a contradiction. However, since we cannot ensure that $I^\psi = \emptyset$, our goal is to find a subset $K_0 \subseteq [t]$ with $|K_0| = k_0$ (see (25)), with $[K_0]^3 \cap I^\psi = \emptyset$ and $G^\psi \cap [K_0]^3$ large. We now produce this set K_0. For convenience, in what follows, we define the following set:

$$\mathrm{K}_{\mathrm{good},\,G^\psi} = \left\{K \in [t]^{k_0} \colon |G^\psi \cap [K]^3| \geq \mathrm{ex}_{\mathrm{ind}}(k_0,\mathbb{F}) + \mu\binom{k_0}{3}\right\}. \tag{75}$$

We now state the following claim, whose proof is postponed to Section 4.2.5.

Claim 34. *The inequality in* (74) *implies that*

$$\left|\mathrm{K}_{\mathrm{good},\,G^{\psi}}\right| \geq \mu^3 \binom{t}{k_0}.$$

We continue with another claim. Once again, for convenience, we use the following notation analogous to (75). Define

$$\mathrm{K}_{\mathrm{good},\,I^{\psi}} = \left\{K \in [t]^{k_0} \colon [K]^3 \cap I^{\psi} = \emptyset\right\}.$$

Our claim is then as follows.

Claim 35.

$$\left|\mathrm{K}_{\mathrm{good},\,I^{\psi}}\right| \geq \left(1 - \delta^{1/2}\binom{k_0}{3}\right)\binom{t}{k_0}. \tag{76}$$

Claim 35 is proved in Section 4.2.5. As a result of Claims 34 and 35, we have

$$\left|\mathrm{K}_{\mathrm{good},\,G^{\psi}} \cap \mathrm{K}_{\mathrm{good},I^{\psi}}\right| \geq \left(\mu^3 - \delta^{1/2}\binom{k_0}{3}\right)\binom{t}{k_0} > 0,$$

where the last inequality follows from (28) and (38). Let $K_0 \in [t]^{k_0}$ satisfy that

$$K_0 \in \mathrm{K}_{\mathrm{good},\,G^{\psi}} \cap \mathrm{K}_{\mathrm{good},I^{\psi}}.$$

For M^{ψ}, N^{ψ}, and G^{ψ} given in (73), set

$$\begin{aligned} M^{\psi}_{K_0} &= M^{\psi} \cap [K_0]^3, \\ N^{\psi}_{K_0} &= N^{\psi} \cap [K_0]^3, \\ G^{\psi}_{K_0} &= G^{\psi} \cap [K_0]^3. \end{aligned}$$

We note the following properties of $M^{\psi}_{K_0}$, $N^{\psi}_{K_0}$, and $G^{\psi}_{K_0}$.

(I) Recall that G^{ψ}, I^{ψ}, M^{ψ}, and N^{ψ} are each triple systems on vertex set $[t]$. Consequently, $G^{\psi}_{K_0}$, $M^{\psi}_{K_0}$, and $N^{\psi}_{K_0}$ are triple systems with vertex set K_0. Moreover, recall that $G^{\psi} \cup I^{\psi} \cup M^{\psi} \cup N^{\psi} = [t]^3$ is a partition of $[t]^3$. Since $I^{\psi} \cap [K_0]^3 = \emptyset$, we infer that $G^{\psi}_{K_0} \cup M^{\psi}_{K_0} \cup N^{\psi}_{K_0} = [K_0]^3$ is a partition of $[K_0]^3$.

(II) Since $K_0 \in \mathrm{K}_{\mathrm{good},\,G^{\psi}}$, the system $G^{\psi}_{K_0}$ satisfies

$$|G^{\psi}_{K_0}| = |G^{\psi} \cap [K_0]^3| \geq \mathrm{ex}_{\mathrm{ind}}(k_0, \mathbb{F}) + \mu\binom{k_0}{3} > \mathrm{ex}_{\mathrm{ind}}(k_0, \mathbb{F}).$$

(III) Following our observations in (I) and (II), since $G^{\psi}_{K_0} \cup M^{\psi}_{K_0} \cup N^{\psi}_{K_0} = [K_0]^3$ is a partition of $[K_0]^3$ and $|G^{\psi}_{K_0}| > \mathrm{ex}_{\mathrm{ind}}(k_0, \mathbb{F})$, it follows directly from the definition of $\mathrm{ex}_{\mathrm{ind}}(k_0, \mathbb{F})$ that there exists $\widetilde{G^{\psi}_{K_0}} \subseteq G^{\psi}_{K_0}$ and $\mathcal{F} \in \mathbb{F}$ so that $\mathcal{F}$ is an induced subhypergraph of $\widetilde{G^{\psi}_{K_0}} \cup M^{\psi}_{K_0}$.

We show that, as a result of the conclusion in (III), the same hypergraph $\mathcal{F}$ must also appear as an induced subhypergraph of $\mathcal{H}$, which will be a contradiction. Before showing this, however, we expand our observation in (III).

Let $\mathcal{F}_0$ be the copy of $\mathcal{F}$ induced in $\widetilde{G^{\psi}_{K_0}} \cup M^{\psi}_{K_0}$. For simplicity of notation, we suppose $V(\mathcal{F}_0) = [f_0] = \{1, \ldots, f_0\}$. We observe the following:

(IV) Each $1 \leq i \leq f_0$ corresponds to a vertex class V_i of our (δ, r)-regular, $(l, t, \gamma, \varepsilon)$-equitable partition $\mathbf{P}$. Moreover, every $1 \leq i < j \leq f_0$ corresponds to a bipartite graph $P^{ij} = P^{ij}_{\alpha}$ with bipartition $V_i \cup V_j$ in our partition $\mathbf{P}$, where $\alpha = \psi(\{i, j\}) > 0$ for the function $\psi\colon [t]^2 \to [l]$ given by Proposition 33. Finally, every $1 \leq i < j < k \leq f_0$ corresponds to a triad $P^{ij} \cup P^{jk} \cup P^{ik}$ of our partition $\mathbf{P}$.

(V) Since $[f_0]^3 \cap I^{\psi} = \emptyset$, each $1 \leq i < j \leq f_0$ corresponds to an $(l, \varepsilon, 2)$-cylinder P^{ij} in our partition $\mathbf{P}$. Moreover, since $[f_0]^3 \cap I^{\psi} = \emptyset$, every triad $P^{ij} \cup P^{jk} \cup P^{ik}$ $(1 \leq i < j < k \leq f_0)$ is (δ, r)-regular with respect to $\mathcal{H}$.

(VI) Every $\{i, j, k\} \in \widetilde{G^{\psi}_{K_0}} \cap \mathcal{F}_0$ corresponds to a triad $P^{ij} \cup P^{jk} \cup P^{ik}$ of our partition $\mathbf{P}$ satisfying that $P^{ij} \cup P^{jk} \cup P^{ik}$ is $(\bar{\alpha}, \delta, r)$-regular with respect to $\mathcal{H}$, for some $\bar{\alpha} = \bar{\alpha}(\{i, j, k\})$ with $\alpha_0 \leq \bar{\alpha} \leq 1 - \alpha_0$, where α_0 is given in (29) and (32).

(VII) Every $\{i, j, k\} \in M^{\psi}_{K_0} \cap \mathcal{F}_0$ corresponds to a triad $P = P^{ij} \cup P^{jk} \cup P^{ik}$ of our partition $\mathbf{P}$ such that $\mathcal{H}$ has P-density $d_P(\mathcal{H})$ satisfying $d_P(\mathcal{H}) > 1 - \alpha_0 > \alpha_0$.

(VIII) For every triple $\{i, j, k\} \in \overline{\mathcal{F}_0}$, where $\overline{\mathcal{F}_0} = [f_0]^3 \setminus \mathcal{F}_0$ is the complement of $\mathcal{F}_0$, we have $\{i, j, k\} \in N^{\psi}_{K_0} \cup G^{\psi}_{K_0}$. Indeed, $\mathcal{F}_0$ is *induced* in $\widetilde{G^{\psi}_{K_0}} \cup M^{\psi}_{K_0}$ and hence, in particular, $\overline{\mathcal{F}_0} \cap M^{\psi}_{K_0} = \emptyset$. We distinguish between two cases.

(VIII)(a) If $\{i, j, k\} \in G^{\psi}_{K_0}$, then we conclude, as in (VI), that the triad $P^{ij} \cup P^{jk} \cup P^{ik}$ is $(\bar{\alpha}, \delta, r)$-regular with respect to $\mathcal{H}$, for some $\bar{\alpha} = \bar{\alpha}(\{i, j, k\})$ with $\alpha_0 \leq \bar{\alpha} \leq 1 - \alpha_0$.

(VIII)(b) If $\{i, j, k\} \in N^{\psi}_{K_0}$, then we conclude that $\mathcal{H}$ has P-density $d_P(\mathcal{H})$ satisfying $d_P(\mathcal{H}) < \alpha_0$, where $P = P^{ij} \cup P^{jk} \cup P^{ik}$.

This concludes our observations concerning $G^{\psi}_{K_0}$, $M^{\psi}_{K_0}$, and $N^{\psi}_{K_0}$.

We proceed with the following proposition.

Proposition 36. *There exists a set $X_0 \subseteq V(\mathcal{H})$ and a function $\pi\colon [f_0] \to X_0$ so that π is an isomorphism between $\mathcal{F}_0$ and $\mathcal{H} \cap [X_0]^3$.*

Note that Proposition 36 contradicts the fact that $\mathcal{H} \in \mathrm{Forb}_{\mathrm{ind}}(n, \mathbb{F})$ since π guarantees that $\mathcal{H} \cap [X_0]^3$ is an induced copy of $\mathcal{F}$ in $\mathcal{H}$. Therefore, assumption (72) is indeed false, and Lemma 32 is proved.

In the remaining sections, we prove Propositions 33 and 36 and Claims 34 and 35.

4.2.4. Proof of Proposition 33. The proof of Proposition 33 will be based on the probabilistic method.

Proof of Proposition 33. We begin our proof by defining

$$\varrho = |G| \Big/ \binom{t}{3} l^3 .$$

If $\varrho = 1$, Proposition 33 is easily implied by Fact 4.1 of [15]. Consequently, we may and shall assume that $\varrho < 1$.

We estimate ϱ. Note that it follows from our assumption in (72) that

$$\varrho = |G| \Big/ \binom{t}{3} l^3 > \binom{t}{3}^{-1} \mathrm{ex}_{\mathrm{ind}}(t, \mathbb{F}) + 4\mu \binom{t}{3} > 4\mu. \tag{77}$$

We now define an auxiliary bipartite graph $\Gamma = (V(\Gamma), E(\Gamma))$. To that effect, let $\Psi = \{\psi\colon [t]^2 \to [l]\}$ and $W = [t]^3 \times [l] \times [l] \times [l]$. We have $\Xi \subset W$. Note that

$$|\Psi| = l^{\binom{t}{2}} \tag{78}$$

and

$$|W| = \binom{t}{3} l^3. \tag{79}$$

Define the bipartite graph Γ by setting

$$V(\Gamma) = \Psi \cup W,$$

and

$$E(\Gamma) = \big\{\{\psi, \xi\}\colon \psi \text{ chooses } \xi\big\}.$$

For a vertex $x \in V(\Gamma)$, let $N_\Gamma(x)$ denote the neighbourhood of x in Γ as usual, that is, $N_\Gamma(x) = \{y \in V(\Gamma)\colon \{x, y\} \in E(\Gamma)\}$. Note that for each $\psi \in \Psi$,

$$|N_\Gamma(\psi)| = \binom{t}{3}, \tag{80}$$

and, for each $\xi \in W$,

$$|N_\Gamma(\xi)| = l^{\binom{t}{2} - 3}. \tag{81}$$

For $\psi \in \Psi$, set

$$\begin{aligned} N_{\Gamma,G}(\psi) &= N_\Gamma(\psi) \cap G, \\ N_{\Gamma,I}(\psi) &= N_\Gamma(\psi) \cap I. \end{aligned}$$

In the notation above, Proposition 33 states that there exists $\psi \in \Psi$ such that

$$|N_{\Gamma,G}(\psi)| \geq \frac{|G|}{l^3}\left(1 - \frac{\mu}{2}\right), \tag{82}$$

and

$$|N_{\Gamma,I}(\psi)| \leq \delta^{1/2} \binom{t}{3}. \tag{83}$$

To prove the existence of $\psi \in \Psi$ satisfying (82) and (83), we proceed with the following considerations. Note that the average degrees $\mathrm{Ave}(|N_{\Gamma,G}|) = |\Psi|^{-1} \sum_{\psi \in \Psi} |N_{\Gamma,G}(\psi)|$ and $\mathrm{Ave}(|N_{\Gamma,I}|) = |\Psi|^{-1} \sum_{\psi \in \Psi} |N_{\Gamma,I}(\psi)|$ satisfy

$$\mathrm{Ave}(|N_{\Gamma,G}|) = |\Psi|^{-1} \sum_{\psi \in \Psi} |N_{\Gamma,G}(\psi)| = |\Psi|^{-1} \sum_{\xi \in G} |N_\Gamma(\xi)|$$

and

$$\mathrm{Ave}(|N_{\Gamma,I}|) = |\Psi|^{-1} \sum_{\psi \in \Psi} |N_{\Gamma,I}(\psi)| = |\Psi|^{-1} \sum_{\xi \in I} |N_\Gamma(\xi)|.$$

As a result of (81), we further see that

$$\mathrm{Ave}(|N_{\Gamma,G}|) = \frac{|G|}{l^3} \tag{84}$$

and

$$\mathrm{Ave}(|N_{\Gamma,I}|) = \frac{|I|}{l^3}. \tag{85}$$

Now, set

$$\eta = \frac{\mu}{2} \times \frac{\varrho}{1-\varrho}, \tag{86}$$

and

$$d = 1 + \frac{2}{\eta}. \tag{87}$$

Note that $d > 1$. We shall see that Proposition 33 easily follows from the next fact.

Fact 37. *For $\psi \in \Psi$ chosen independently and uniformly at random, we have*

$$\Pr\left[\psi \text{ satisfies } |N_{\Gamma,G}(\psi)| > \frac{|G|}{l^3}\left(1 - \frac{\mu}{2}\right) \text{ and } |N_{\Gamma,I}(\psi)| \le d\frac{|I|}{l^3}\right] > 0.$$

Before proving Fact 37, we observe that the existence of $\psi \in \Psi$ satisfying $|N_{\Gamma,G}(\psi)| > |G|l^{-3}(1-\mu/2)$ and $|N_{\Gamma,I}(\psi)| \le |I|dl^{-3}$ proves Proposition 33. Indeed, to see that $|N_{\Gamma,I}(\psi)| \le \delta^{1/2}\binom{t}{3}$, consider the following. With $d = 1 + 2/\eta$, we see by (54) that

$$d\frac{|I|}{l^3} \le \left(1 + \frac{2}{\eta}\right) 100\delta \binom{t}{3} = \left(100\delta + \frac{200\delta}{\eta}\right)\binom{t}{3}.$$

With η given in (86) and the inequality in (77), we further see that

$$d\frac{|I|}{l^3} \le \left(100\delta + \frac{100\delta}{\mu^2}\right)\binom{t}{3}.$$

With the inequality in (33), we conclude that

$$d\frac{|I|}{l^3} \le \left(100\delta + \frac{100\delta}{\mu^2}\right)\binom{t}{3} \le \delta^{1/2}\binom{t}{3}.$$

The proof of Proposition 33 is complete. □

We now prove Fact 37.

Proof of Fact 37. Independently and uniformly at random, select $\psi \in \Psi$. Clearly,

$$\Pr\left[|N_{\Gamma,G}(\psi)| < \frac{|G|}{l^3}\left(1 - \frac{\mu}{2}\right) \text{ or } |N_{\Gamma,I}(\psi)| > d\frac{|I|}{l^3}\right]$$
$$\le \Pr\left[|N_{\Gamma,G}(\psi)| < \frac{|G|}{l^3}\left(1 - \frac{\mu}{2}\right)\right] + \Pr\left[|N_{\Gamma,I}(\psi)| > d\frac{|I|}{l^3}\right]. \tag{88}$$

We bound (88) from above, and begin by easily estimating the second term on the right-hand side of this inequality.

We see that

$$\Pr\left[|N_{\Gamma,I}(\psi)| > d\frac{|I|}{l^3}\right] < \frac{1}{d}. \tag{89}$$

Indeed, (85) says that

$$\mathrm{E}\big[|N_{\Gamma,I}(\psi)|\big] = \frac{|I|}{l^3},$$

and hence Markov's inequality tells us that (89) follows.

To bound the first term on the right-hand side of (88), we proceed with the following fact.

Fact 38.

$$\Pr\left[|N_{\Gamma,G}(\psi)| < \frac{|G|}{l^3}\left(1-\frac{\mu}{2}\right)\right] < \frac{1}{1+\eta}.$$

We now continue our proof of Fact 37 assuming Fact 38. For the randomly chosen $\psi \in \Psi$, we infer from (88), (89), and (98) that

$$\Pr\left[|N_{\Gamma,G}(\psi)| < \frac{|G|}{l^3}\left(1-\frac{\mu}{2}\right) \text{ or } |N_{\Gamma,I}(\psi)| > d\frac{|I|}{l^3}\right] < \frac{1}{d} + \frac{1}{1+\eta}. \tag{90}$$

However, as defined in (87), $d = 1 + 2/\eta$, thus

$$\frac{1}{d} + \frac{1}{1+\eta} = \frac{\eta}{2+\eta} + \frac{1}{1+\eta} < 1.$$

Thus our proof of Fact 37 is complete. □

We now have to prove Fact 38.

Proof of Fact 38. For $\psi \in \Psi$, we infer from (80) that

$$\big|N_{\Gamma,G}(\psi)\big| + \big|N_{\Gamma}(\psi) \setminus N_{\Gamma,G}(\psi)\big| = \big|N_{\Gamma}(\psi)\big| = \binom{t}{3}. \tag{91}$$

Note that (84) says that for $\psi \in \Psi$ randomly chosen,

$$\mathrm{E}\big[|N_{\Gamma,G}(\psi)|\big] = \frac{|G|}{l^3}.$$

From (91), we thus infer that

$$\begin{aligned} \mathrm{E}\big[|N_{\Gamma}(\psi) \setminus N_{\Gamma,G}(\psi)|\big] &= \binom{t}{3} - \mathrm{E}\big[|N_{\Gamma,G}(\psi)|\big], \\ &= \binom{t}{3} - \frac{|G|}{l^3}. \end{aligned} \tag{92}$$

Now, for $\psi \in \Psi$, if

$$|N_{\Gamma,G}(\psi)| < \frac{|G|}{l^3}\left(1-\frac{\mu}{2}\right), \tag{93}$$

it would then follow from (91) that

$$\left|N_\Gamma(\psi) \setminus N_{\Gamma,G}(\psi)\right| > \binom{t}{3} - \frac{|G|}{l^3}\left(1 - \frac{\mu}{2}\right). \tag{94}$$

One can calculate that with $\eta = (\mu/2)(\varrho/(1-\varrho))$, we have

$$\binom{t}{3} - \frac{|G|}{l^3}\left(1 - \frac{\mu}{2}\right) \geq (1+\eta)\left(\binom{t}{3} - \frac{|G|}{l^3}\right).$$

Thus, from (92), if $\psi \in \Psi$ is randomly chosen and satisfies (93), then

$$\left|N_\Gamma(\psi) \setminus N_{\Gamma,G}(\psi)\right| > \binom{t}{3} - \frac{|G|}{l^3}\left(1 - \frac{\mu}{2}\right) \geq (1+\eta)E\big[|N_\Gamma(\psi) \setminus N_{\Gamma,G}(\psi)|\big]. \tag{95}$$

Thus, we infer from (93), (94), and (95) that

$$\Pr\left[|N_{\Gamma,G}(\psi)| < \frac{|G|}{l^3}\left(1 - \frac{\mu}{2}\right)\right] \leq \Pr\left[\left|N_\Gamma(\psi) \setminus N_{\Gamma,G}(\psi)\right| > (1+\eta)E\big[\left|N_\Gamma(\psi) \setminus N_{\Gamma,G}(\psi)\right|\big]\right]. \tag{96}$$

Using Markov's inequality, we see that

$$\Pr\left[\left|N_\Gamma(\psi) \setminus N_{\Gamma,G}(\psi)\right| > (1+\eta)E\big[\left|N_\Gamma(\psi) \setminus N_{\Gamma,G}(\psi)\right|\big]\right] < \frac{1}{1+\eta}. \tag{97}$$

We infer from (96) and (97) that

$$\Pr\left[|N_{\Gamma,G}(\psi)| < \frac{|G|}{l^3}\left(1 - \frac{\mu}{2}\right)\right] < \frac{1}{1+\eta}, \tag{98}$$

and thus the proof of Fact 38 is complete. □

4.2.5. Proof of Claims 34 and 35. In this section we prove the two claims that we used in the proof of Lemma 32. We start with Claim 34.

Proof of Claim 34. On the contrary, assume

$$|\mathrm{K}_{\mathrm{good},\,G^\psi}| < \mu^3\binom{t}{k_0}. \tag{99}$$

Define an auxiliary bipartite graph $\Lambda = (V(\Lambda), E(\Lambda))$ with bipartition $V(\Lambda) = [t]^{k_0} \cup [t]^3$ and edge set $E(\Lambda) = \big\{\{K, Y\}\colon K \in [t]^{k_0},\ Y \in [K]^3 \cap G^\psi\big\}$. Then

$$|G^\psi|\binom{t-3}{k_0-3} = |E(\Lambda)| = \sum_{K \in [t]^{k_0}} \deg_\Lambda(K). \tag{100}$$

By (74), we see that

$$|G^\psi|\binom{t-3}{k_0-3} > \left(\mathrm{ex}_{\mathrm{ind}}(t, \mathbb{F}) + 2\mu\binom{t}{3}\right)\binom{t-3}{k_0-3}. \tag{101}$$

On the other hand, our assumption in (99) says that more than $(1-\mu^3)\binom{t}{k_0}$ sets $K \in [t]^{k_0}$ satisfy

$$\deg_\Lambda(K) < \mathrm{ex}_{\mathrm{ind}}(k_0, \mathbb{F}) + \mu\binom{k_0}{3}. \tag{102}$$

Applying (101) and (102) to (100), we conclude

$$\begin{aligned}\binom{t-3}{k_0-3}&\left(\mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F})+2\mu\binom{t}{3}\right)\\ &<(1-\mu^3)\binom{t}{k_0}\left(\mathrm{ex}_{\mathrm{ind}}(k_0,\mathbb{F})+\mu\binom{k_0}{3}\right)+\mu^3\binom{t}{k_0}\binom{k_0}{3}\\ &\le\binom{t}{k_0}\left(\mathrm{ex}_{\mathrm{ind}}(k_0,\mathbb{F})+\mu\binom{k_0}{3}\right)+\mu^3\binom{t}{k_0}\binom{k_0}{3}.\end{aligned}$$

Dividing through by $\binom{t}{k_0}\binom{k_0}{3}$, we obtain

$$\binom{t}{3}^{-1}\mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F})+2\mu<\binom{k_0}{3}^{-1}\mathrm{ex}_{\mathrm{ind}}(k_0,\mathbb{F})+\mu+\mu^3,$$

or, equivalently,

$$\mu<\binom{k_0}{3}^{-1}\mathrm{ex}_{\mathrm{ind}}(k_0,\mathbb{F})-\binom{t}{3}^{-1}\mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F})+\mu^3. \tag{103}$$

Note that with t_0 and k_0 defined in (25) and (39), and with $t\ge t_0$, we have

$$\binom{k_0}{3}^{-1}\mathrm{ex}_{\mathrm{ind}}(k_0,\mathbb{F})-\binom{t}{3}^{-1}\mathrm{ex}_{\mathrm{ind}}(t,\mathbb{F})<\tau. \tag{104}$$

Employing (104) in (103), we obtain

$$\mu<\tau+\mu^3.$$

With $\tau<\mu^3$ (see (23)), we further conclude that

$$\mu<2\mu^3,$$

which contradicts $\mu<1/2$ (see (23)). Thus, Claim 34 is proved. □

Proof of Claim 35. For each $\{i,j,k\}_{\alpha\beta\gamma}\in I^\psi$, there are at most $\binom{t-3}{k_0-3}$ sets $K\in[t]^{k_0}$ with $\{i,j,k\}_{\alpha\beta\gamma}\in[K]^3$. Since $|I^\psi|<\delta^{1/2}\binom{t}{3}$, we have

$$\begin{aligned}\left|\left\{K\in[t]^{k_0}\colon[K]^3\cap I^\psi\ne\emptyset\right\}\right| &\le |I^\psi|\binom{t-3}{k_0-3},\\ &< \delta^{1/2}\binom{t}{3}\binom{t-3}{k_0-3},\\ &= \delta^{1/2}\binom{k_0}{3}\binom{t}{k_0},\end{aligned}$$

which implies (76) and hence Claim 35 is proved. □

4.2.6. Proof of Proposition 36. We first produce the promised set X_0. To that end, define the f_0-partite graph

$$P_{f_0}=\bigcup_{1\le i<j\le f_0}P^{ij}.$$

Note that P_{f_0} has vertex set $V_1 \cup \ldots \cup V_{f_0}$. Note that, because of (V), our graph P_{f_0} is an (l, ε, f_0)-cylinder.

In what follows, we shall work with triples $\vartheta = \{i,j,k\}$, where $1 \leq i < j < k \leq f_0$. Let $\Theta = \big\{\vartheta = \{i,j,k\}\colon 1 \leq i < j < k \leq f_0\big\}$ be the set of such triples. For each $\vartheta = \{i,j,k\} \in \Theta$, we have the triad $P_\vartheta = P^{ij} \cup P^{jk} \cup P^{ik}$. Moreover, for each $\vartheta = \{i,j,k\} \in \Theta$, define the triple system

$$\mathcal{D}^\vartheta = \begin{cases} \mathcal{H} \cap \mathcal{K}_3(P_\vartheta) & \text{if } \vartheta \in \mathcal{F}_0, \\ \mathcal{K}_3(P_\vartheta) \setminus \mathcal{H} & \text{if } \vartheta \notin \mathcal{F}_0. \end{cases}$$

Set

$$\mathcal{D} = \bigcup_{\vartheta \in \Theta} \mathcal{D}^\vartheta.$$

Note that $\mathcal{D}$ has vertex set $V_1 \cup \ldots \cup V_{f_0}$. We now make the following observations about $\mathcal{D}$ and P_{f_0}.

(a) Because of (V), for every $\vartheta = \{i,j,k\} \in \mathcal{F}_0$, the 3-partite hypergraph $\mathcal{D}^\vartheta$ is (δ, r)-regular with respect to P_ϑ. Additionally, as $\mathcal{H}$ is (δ, r)-regular with respect to every P_ϑ $(\vartheta \in \Theta)$, it follows that $\mathcal{K}_3(P_\vartheta) \setminus \mathcal{H}$ is also (δ, r)-regular with respect to P_ϑ. Consequently, for every $\vartheta \in \Theta$, the system $\mathcal{D}^\vartheta$ is (δ, r)-regular with respect to P_ϑ.

(b) Observe that, for every $\vartheta \notin \mathcal{F}_0$, we have

$$d_{P_\vartheta}(\mathcal{D}^\vartheta) = 1 - d_{P_\vartheta}(\mathcal{H}).$$

(c) Because of (VI) and (VII), if $\vartheta \in \mathcal{F}_0$, then $\mathcal{D}^\vartheta$ is $(\bar{\alpha}, \delta, r)$-regular with respect to P_ϑ, where $\bar{\alpha} = \bar{\alpha}(\vartheta) \geq \alpha_0$.

(d) Suppose that $\vartheta \in \Theta \setminus \mathcal{F}_0$. From (VIII) we see that $\vartheta \in G^\psi_{K_0} \cup N^\psi_{K_0}$. From (VIII)(a), we have that if $\vartheta \in G^\psi_{K_0}$, then $d_{P_\vartheta}(\mathcal{H}) \leq 1 - \alpha_0$. If $\vartheta \in N^\psi_{K_0}$, then $d_{P_\vartheta}(\mathcal{H}) < \alpha_0$. Using ($b$), we conclude in either case that $d_{P_\vartheta}(\mathcal{D}^\vartheta) = 1 - d_{P_\vartheta}(\mathcal{H}) \geq \alpha_0$. Thus, for $\vartheta \in \Theta \setminus \mathcal{F}_0$, we have that $\mathcal{D}^\vartheta$ is $(\bar{\alpha}, \delta, r)$-regular with respect to P_ϑ, where $\bar{\alpha} = \bar{\alpha}(\vartheta) \geq \alpha_0$.

(e) Combining our observations in (c) and (d), for all ϑ, the system $\mathcal{D}^\vartheta$ is $(\bar{\alpha}, \delta, r)$-regular with respect to P_ϑ, where $\bar{\alpha} = \bar{\alpha}(\vartheta) \geq \alpha_0$.

(f) Owing to the inequalities in (37) and (42), we have $\delta \leq \delta_{31}(f_0, \alpha_0)$ and $r \geq r_{31}(f_0, \alpha_0, \delta, l)$, where $\delta_{31}(f_0, \alpha_0)$ verified the applicability of Lemma 31 for the parameters f_0 and α_0 and $r_{31}(f_0, \alpha_0, \delta, l)$ verified the applicability of Lemma 31 for the parameters f_0, α_0, δ, and l.

(g) P_{f_0} is an (l, ε, f_0)-cylinder. Here, because of (43), we have $\varepsilon \leq \varepsilon_{31}(f_0, \alpha_0, \delta, l, r)$, where $\varepsilon_{31}(f_0, \alpha_0, \delta, l, r)$ verified the applicability of Lemma 31 for the parameters f_0, α_0, δ, l, and r.

(h) By observations (a)–(g) above, we see that $\mathcal{D}$ and P_{f_0} satisfy the hypotheses of Lemma 31. As a result, we conclude that $\mathcal{D}$ contains a copy of $K^{(3)}_{f_0}$.

We are nearly finished with the proof of Proposition 36. Let X_0 be a vertex set inducing a copy of $K^{(3)}_{f_0}$ in $\mathcal{D}$. We now show that X_0 induces a copy of $\mathcal{F}$ in $\mathcal{H}$. Indeed, define a function $\pi\colon [f_0] \to X_0$ by letting $\pi(i)$ be the unique element of $V_i \cap X_0$, for each $i \in [f_0]$. We proceed with the following claim.

Claim 39. *The map $\pi\colon [f_0] \to X_0$ is an isomorphism between $\mathcal{F}_0$ and $\mathcal{H} \cap [X_0]^3$.*

Proof. Clearly π is a bijection. We show that $\vartheta = \{i,j,k\} \in \mathcal{F}_0$ if and only if $\pi(\vartheta) = \{\pi(i),\pi(j),\pi(k)\} \in \mathcal{H} \cap [X_0]^3$. Let $\vartheta \in \Theta$ be given. As X_0 spans a copy of $K_{f_0}^{(3)}$ in $\mathcal{D}$, clearly, $\pi(\vartheta) \in \mathcal{D}^\vartheta$. We consider the following two cases.

Case 1. $\vartheta \in \mathcal{F}_0$

In this case, by the definition of D^ϑ, we have $D^\vartheta = \mathcal{H} \cap \mathcal{K}_3(P_\vartheta)$. As $\pi(\vartheta) \in \mathcal{D}^\vartheta$, we have $\pi(\vartheta) \in \mathcal{H}$.

Case 2. $\vartheta \in \Theta \setminus \mathcal{F}_0$

In this case, by the definition of D^ϑ, we have $D^\vartheta = \mathcal{K}_3(P_\vartheta) \setminus \mathcal{H}$. As $\pi(\vartheta) \in \mathcal{D}^\vartheta$, we have $\pi(\vartheta) \notin \mathcal{H}$.

In view of the above, the proof of Claim 39 is complete. □

This finishes our proof of Proposition 36.

5. Concluding remarks

We believe that the investigation of the parameter $\mathrm{ex}_{\mathrm{ind}}(n,\mathcal{F})$ may be of independent interest. As observed before, the determination of $\mathrm{ex}_{\mathrm{ind}}(n,\mathcal{F})$ for general hypergraphs $\mathcal{F}$ will not be simple, as this problem is more general than that of determining Turán numbers for complete hypergraphs. It would be interesting to study the behaviour of $\mathrm{ex}_{\mathrm{ind}}(n,\mathcal{F})$ in greater depth, further clarifying the similarities and differences between $\mathrm{ex}_{\mathrm{ind}}(n,\mathcal{F})$ and the classical Turán numbers $\mathrm{ex}(n,\mathcal{F})$. Let us give an example.

Recall that we defined the sets $L_{<\infty}$ and $L_{<\infty}^{\mathrm{ind}}$ in Section 2, and that these sets are not well-ordered. Somewhat unexpectedly, the set of 'left accumulation points' of these sets coincide (we say that α is a *left accumulation point* of a set $Z \subset \mathbb{R}$ if one may find a strictly decreasing sequence $(\alpha_t)_{t=1}^\infty$ in Z with $\alpha_t \searrow \alpha$). We hope to come back to this and related results in the near future.

The proof of our main result in this paper, Theorem 1, is based in a crucial manner on the Regularity Lemma from [11] and on the Counting Lemma from [16]. Even with these results at hand, the proof of Theorem 1 is fairly elaborate, at least from the technical point of view, because the basic objects involved in the Regularity Lemma and in the Counting Lemma are already somewhat technical. We believe that simplifying these lemmas to the point of making applications such as the one presented in this paper straightforward would be of great interest.

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