CONSTRUCTIVE PACKINGS OF TRIPLE SYSTEMS

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Abstract. Let $\mathcal{F}_0$ and $\mathcal{H}$ be a pair of k-graphs (written F_0 and H, respectively, when $k = 2$). An $\mathcal{F}_0$-packing of $\mathcal{H}$ is a family $\mathscr{F}$ of pairwise edge-disjoint copies of $\mathcal{F}_0$ in $\mathcal{H}$. Let $\nu_{\mathcal{F}_0}(\mathcal{H})$ denote the maximum size $|\mathscr{F}|$ of an $\mathcal{F}_0$-packing $\mathscr{F}$ of $\mathcal{H}$. Already in the case of graphs ($k = 2$), Dor and Tarsi proved that computing $\nu_{F_0}(H)$ is NP-hard for every fixed graph F_0 having a component with three or more edges. On the other hand, Rödl et al. (see primarily [19]) proved that, for any fixed k-graph $\mathcal{F}_0$, the parameter $\nu_{\mathcal{F}_0}(\mathcal{H})$ can be approximated within an error of $o(|V(\mathcal{H})|^k)$ in time polynomial in $|V(\mathcal{H})|$. In particular, a foundational result of Haxell and Rödl [11] for graphs ($k = 2$) constructs, for every fixed graph F_0 and for every given graph H, an F_0-packing $\mathscr{F}$ of size $|\mathscr{F}| \geq \nu_{F_0}(H) - o(|V(H)|^2)$ in time polynomial in $|V(H)|$.

In this paper, we extend the result of Haxell and Rödl to $k = 3$. In particular, for a fixed 3-graph $\mathcal{F}_0$, we establish an algorithm which, for all $\zeta > 0$ and for every given 3-graph $\mathcal{H}$, constructs an $\mathcal{F}_0$-packing $\mathscr{F}$ of $\mathcal{H}$ of size $|\mathscr{F}| \geq \nu_{\mathcal{F}_0}(\mathcal{H}) - \zeta|V(\mathcal{H})|^3$ in time polynomial in $|V(\mathcal{H})|$. Our approach is based on that of Haxell and Rödl, and uses hypergraph regularity tools of them and the author from earlier papers, together with some details proven here.

1. Introduction

Let $\mathcal{F}_0$ and $\mathcal{H}$ be a pair of k-uniform hypergraphs (k-graphs for short, written F_0 and H, respectively, when $k = 2$). An $\mathcal{F}_0$-*packing* of $\mathcal{H}$ is a family $\mathscr{F}$ of pairwise edge-disjoint copies of $\mathcal{F}_0$ in $\mathcal{H}$. Let $\nu_{\mathcal{F}_0}(\mathcal{H})$ denote the maximum size $|\mathscr{F}|$ of an $\mathcal{F}_0$-packing $\mathscr{F}$ of $\mathcal{H}$. (By definition, every $\mathcal{F}_0$-packing $\mathscr{F}$ of $\mathcal{H}$ has $|\mathscr{F}| \leq |\mathcal{H}|/|\mathcal{F}_0| = O(|V(\mathcal{H})|^k)$ many elements.) Already in the case of graphs, the parameter $\nu_{F_0}(H)$ is NP-hard to compute for every fixed graph F_0 having a component with three or more edges (see Dor and Tarsi [5]). However, Rödl et al. [19, 11, 10] (see Theorem 1.4 below) proved that, for any fixed k-graph $\mathcal{F}_0$, the parameter $\nu_{\mathcal{F}_0}(\mathcal{H})$ can be approximated within an error of $o(|V(\mathcal{H})|^k)$ in time polynomial in $|V(\mathcal{H})|$. In the case of graphs ($k = 2$), more is known by the following foundational result of Haxell and Rödl [11].

Theorem 1.1 (Haxell and Rödl [11])**.** *For every graph F_0, and for all $\zeta > 0$, there exists an integer $n_0 = n_0(F_0, \zeta)$ so that, for a given graph H on $n > n_0$ vertices, an F_0-packing $\mathscr{F}$ of size $|\mathscr{F}| \geq \nu_{F_0}(H) - \zeta n^2$ can be constructed in time polynomial in n.*

Note that Theorem 1.1 also holds for $n \leq n_0$ by exhaustive search (and in constant time).

Theorem 1.1 was extended to linear k-graphs $\mathcal{F}_0$, where a k-graph $\mathcal{F}_0$ is *linear* if every pair of its edges overlaps in at most one vertex. (Every simple graph F_0 is a linear 2-graph.)

Theorem 1.2 (Dizona and Nagle [4])**.** *For every $k \geq 2$, for every linear k-graph $\mathcal{F}_0$, and for all $\zeta > 0$, there exists an integer $n_0 = n_0(k, \mathcal{F}_0, \zeta)$ so that, for a given k-graph $\mathcal{H}$ on $n > n_0$ vertices, an $\mathcal{F}_0$-packing $\mathscr{F}$ of size $|\mathscr{F}| \geq \nu_{\mathcal{F}_0}(\mathcal{H}) - \zeta n^k$ can be constructed in time polynomial in n.*

In this paper, we prove an analogue of Theorems 1.1 and 1.2 for $k = 3$ and arbitrary 3-graphs $\mathcal{F}_0$.

Theorem 1.3 (Main result)**.** *For every 3-graph $\mathcal{F}_0$, and for all $\zeta > 0$, there exists an integer $n_0 = n_0(\mathcal{F}_0, \zeta)$ so that, for a given 3-graph $\mathcal{H}$ on $n > n_0$ vertices, an $\mathcal{F}_0$-packing $\mathscr{F}$ of size $|\mathscr{F}| \geq \nu_{\mathcal{F}_0}(\mathcal{H}) - \zeta n^3$ can be constructed in time polynomial in n.*

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Theorem 1.3 was thought to be possible by Haxell, Rödl and the author in [9], in light of the tools proven there, at least if one followed the approach of Haxell and Rödl for Theorem 1.1. (We outline this approach momentarily.) This paper is the corresponding realization (see the Acknowledgment at the end of this Introduction). It was also anticipated in [9] that further details would need to be developed (see Remark 1.6 below), which this paper considers. We believe that some of the auxiliary tools proven here could be of use in other contexts, and may be of some independent interest.

The proofs of Theorems 1.1–1.4 all rely on two main ingredients: *fractional packings* and *regularity methods*. Fractional packings are much easier to describe than their regularity counterparts, so we will begin here (and we will also present Theorem 1.4, which is in terms of fractional packings). Afterward, we will outline the regularity tools employed in the proof of Theorem 1.1, which we hope will somewhat motivate upcoming parallels in this paper. We also mention that Theorems 1.1–1.3 can all be proven because graphs, linear hypergraphs, and 3-graphs, resp., were (until quite recently [17, 18]) precisely where regularity methods had known algorithms.

1.1. **Fractional packings.** For k-graphs $\mathcal{F}_0$ and $\mathcal{H}$, let $\binom{\mathcal{H}}{\mathcal{F}_0}$ denote the family of all copies of $\mathcal{F}_0$ in $\mathcal{H}$. For an edge $e \in \mathcal{H}$, let $\binom{\mathcal{H}}{\mathcal{F}_0}_e$ denote the family of copies $\mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}$ containing the edge e. In this notation, an $\mathcal{F}_0$-packing of $\mathcal{H}$ is a family $\mathscr{F} \subseteq \binom{\mathcal{H}}{\mathcal{F}_0}$ satisfying that, for each fixed edge $e \in \mathcal{H}$,

$$\left| \mathscr{F} \cap \binom{\mathcal{H}}{\mathcal{F}_0}_e \right| \leq 1. \tag{1}$$

Fractional $\mathcal{F}_0$-packings generalize $\mathcal{F}_0$-packings, and can be defined when $\mathcal{H}$ has edge-weights. For a set V and a weight function $\omega : \binom{V}{k} \to [0,1]$, write $\mathcal{H} = \omega^{-1}(0,1]$ as the family of edges of positive weight, and write $\mathcal{H}^\omega = \{(e, \omega(e)) : e \in \mathcal{H}\}$ as the family of such edges together with their weights. (In the unweighted case when $\omega : \binom{V}{k} \to \{0,1\}$, we identify $\mathcal{H} = \mathcal{H}^\omega$.) A function $\psi : \binom{\mathcal{H}}{\mathcal{F}_0} \to [0,1]$ is a *fractional $\mathcal{F}_0$-packing* of $\mathcal{H}^\omega$ if, for each edge $e \in \mathcal{H}$,

$$\sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_e \right\} \leq \omega(e). \tag{2}$$

Define $|\psi| = \sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0} \right\}$ to be the *size* of ψ. Define $\nu^*_{\mathcal{F}_0}(\mathcal{H}^\omega)$ to be the maximum size $|\psi|$ of a fractional $\mathcal{F}_0$-packing of $\mathcal{H}^\omega$.

To motivate Theorem 1.4 below, we relate the parameters $\nu_{\mathcal{F}_0}(\mathcal{H})$ and $\nu^*_{\mathcal{F}_0}(\mathcal{H})$ (for a fixed $\mathcal{F}_0$ and an unweighted $\mathcal{H} = \mathcal{H}^\omega$), in terms of relative size and relative complexity. First, consider an optimal $\mathcal{F}_0$-packing $\mathscr{F}$ of $\mathcal{H}$. Then the characteristic function $\chi_{\mathscr{F}} : \binom{\mathcal{H}}{\mathcal{F}_0} \to \{0,1\}$ of $\mathscr{F}$ is a fractional $\mathcal{F}_0$-packing of $\mathcal{H}$ (cf. (1) and (2)), and so $\nu_{\mathcal{F}_0}(\mathcal{H}) \leq \nu^*_{\mathcal{F}_0}(\mathcal{H})$. Second, while computing $\nu_{\mathcal{F}_0}(\mathcal{H})$ is known to be a difficult problem, constructing a fractional $\mathcal{F}_0$-packing ψ of $\mathcal{H}$ with optimal size $|\psi| = \nu^*_{\mathcal{F}_0}(\mathcal{H})$ is a linear programming problem requiring time only polynomial in $n = |V(\mathcal{H})|$. Theorem 1.4 is now motivated.

Theorem 1.4 (Rödl, Schacht, Siggers, Tokushige [19])**.** *For every $k \geq 2$, for every k-graph $\mathcal{F}_0$, and for all $\zeta > 0$, there exists an integer $n_0 = n_0(k, \mathcal{F}_0, \zeta)$ so that, for every k-graph $\mathcal{H}$ on $n > n_0$ vertices,*

$$\nu^*_{\mathcal{F}_0}(\mathcal{H}) \leq \nu_{\mathcal{F}_0}(\mathcal{H}) + \zeta n^k.$$

Thus, $\nu_{\mathcal{F}_0}(\mathcal{H})$ can be approximated within an error of ζn^k in time polynomial in n.

Various cases of Theorem 1.4 had been earlier considered. Haxell and Rödl initiated Theorem 1.4 when, for graphs ($k = 2$), they proved Theorem 1.4 in the stronger form of Theorem 1.1. Yuster [23] gave an alternative proof of Theorem 1.4 for graphs ($k = 2$) which allowed F_0 to be replaced by a fixed family of graphs. For $k = 3$, Theorem 1.4 was proven by Haxell, Nagle and Rödl [9].

1.2. **Regularity and the proof of Theorem 1.1.** We outline the regularity tools in the proof of Theorem 1.1, and outline the proof of Theorem 1.1 in the case that $F_0 = K_3$ is the triangle.

The main tool from [11] is the Szemerédi Regularity Lemma. Let G be a bipartite graph with vertex bipartition $V(G) = X \cup Y$. For non-empty $X' \subseteq X$ and $Y' \subseteq Y$, the *density* of G w.r.t. X' and Y'

is $d_G(X', Y') = |G[X', Y']|/(|X'||Y'|)$, where $G[X', Y']$ is the subgraph of G induced on $X' \cup Y'$. For $d \geq 0$ and $\varepsilon > 0$, we say that G is *(d, ε)-regular* if, for every $X' \subseteq X$, where $|X'| > \varepsilon|X|$, and for every $Y' \subseteq Y$, where $|Y'| > \varepsilon|Y|$, we have $|d_G(X', Y') - d| < \varepsilon$. For $\varepsilon > 0$, we say that G is *ε-regular* if it is (d, ε)-regular for some $d \geq 0$.

Theorem 1.5 (Szemerédi's Regularity Lemma [20, 21])**.** *For all $\varepsilon > 0$, and for every integer $t_0 \geq 1$, there exist integers $T_0 = T_0(\varepsilon, t_0)$ and $N_0 = N_0(\varepsilon, t_0)$ so that every graph H on $n > N_0$ vertices admits a vertex partition $\Pi : V(H) = V_0 \cup V_1 \cup \cdots \cup V_t$, for some $t_0 \leq t \leq T_0$, satisfying the following conditions:*

1. *Π is* t-equitable*, meaning $|V_1| = \cdots = |V_t| \stackrel{\text{def}}{=} m \geq \lfloor n/t \rfloor$;*
2. *Π is* ε-regular*, meaning for all but $\varepsilon\binom{t}{2}$ pairs V_i, V_j, $1 \leq i < j \leq t$, $H[V_i, V_j]$ is ε-regular.*

In the proof of Theorem 1.1, one needs a *constructive* version of Szemerédi's Regularity Lemma, which was established by Alon, Duke, Lefmann, Rödl and Yuster [1]. Their result guarantees that the partition Π in Theorem 1.5 can be constructed in time $O(M(n))$, where $M(n) = O(n^{2.3727})$ (cf. [22]) is the time required to multiply two $n \times n$ binary matrices over the integers. Kohayakawa, Rödl and Thoma [13] later improved the constructive Regularity Lemma to run in time $O(n^2)$. (Either constructive version of the Regularity Lemma serves our purposes here.)

Proof-sketch of Theorem 1.1 for triangles. Fix $k = 2$ and $F_0 = K_3$.

Input. Let $\zeta > 0$ be given. It is possible to fix $\zeta', \beta, \zeta'', \varepsilon', \varepsilon > 0$ and $t_0, T_0, n \in \mathbb{N}$ satisfying

$$\zeta > \zeta' = \frac{\zeta}{4} \gg \beta \gg \zeta'' \gg \varepsilon' \gg \varepsilon = \frac{1}{t_0} \gg \frac{1}{T_0} \ggg \frac{1}{n}, \tag{3}$$

in such a way as to support all details in the sketch below. (In particular, the notation $x \ll y, z, \ldots$ appearing in (3) means that $x > 0$ can be chosen as a sufficiently small function of $y, z, \ldots > 0$ to satisfy an upcoming list of computations involving $x, y, z, \ldots$. The notation $1/n \lll \ldots$ appearing in (3) means the integer n will be chosen sufficiently large with respect to all constants in (3).) Let H be a graph on n vertices. We build, in time polynomial in n, a triangle packing $\mathscr{T}$ of H of size $|\mathscr{T}| \geq \nu^*_{K_3}(H) - \zeta n^2 \geq \nu_{K_3}(H) - \zeta n^2$.

Step 1: Applying the Regularity Lemma. With $\varepsilon = 1/t_0$ from (3), use the algorithm of Kohayakawa et al. [13] (cf. [1]) to construct, in time $O(n^2)$, an ε-regular, t-equitable partition $\Pi : V(H) = V_0 \cup V_1 \cup \cdots \cup V_t$, where $t_0 \leq t \leq T_0 = O(1)$. (Record also, in time $O(n^2)$, the densities $d(V_i, V_j)$, $1 \leq i < j \leq t$, and the pairs (V_i, V_j), $1 \leq i < j \leq t$, which are[1] ε-regular.) Construct, in time $O(t^2) = O(1)$, the corresponding (weighted) cluster graph H_0^ω, as follows. For $\{h, i\} \in \binom{[t]}{2}$, define $\omega(\{h, i\}) = d_H(V_h, V_i)$ if $H[V_h, V_i]$ is ε-regular, and $\omega(\{h, i\}) = 0$ otherwise. Set $H_0 = \omega^{-1}(0, 1]$ and $H_0^\omega = \{(e, \omega(e)) : e \in H_0\}$. A standard argument shows that

$$\nu^*_{K_3}(H_0^\omega) \geq \frac{\nu^*_{K_3}(H)}{m^2} - \zeta' t^2, \tag{4}$$

where $m = |V_1| = \cdots = |V_t|$. We now pause to reveal a few points of strategy.

Pause (strategy). A main tool for building $\mathscr{T}$ is the following so-called *Packing Lemma* (see Lemma 5 in [11]). Suppose that for some $1 \leq h < i < j \leq t$, each of $H[V_h, V_i]$, $H[V_i, V_j]$, $H[V_h, V_j]$ is (d, ε)-regular with $d \geq \beta$ (cf. (3)). In time polynomial in m, the Packing Lemma constructs a triangle packing $\mathscr{T}^{hij}$ of $H[V_h, V_i, V_j]$ covering all but $\zeta'' m^2$ edges of $H[V_h, V_i, V_j]$ (cf. (3)), in which case

$$\left|\mathscr{T}^{hij}\right| \geq (d - \varepsilon - \zeta'')m^2 \geq (d - 2\zeta'')m^2 \stackrel{(3)}{\geq} \left(1 - \sqrt{\zeta''}\right) dm^2 \quad \text{(recall } d \geq \beta\text{)}. \tag{5}$$

[1]Recognizing when $H[V_i, V_j]$, $1 \leq i < j \leq t$, is ε-regular is co-NP-complete (see Theorem 2.1 in [1]). However, the algorithms of Alon et al. [1] and Kohayakawa et al. [13] will have already recognized 'most' of the $H[V_i, V_j]$ which are ε-regular (see [1, 13] or Section 1.1 of [9] for details). For the concept of hypergraph regularity that we use in this paper, we will always be able to recognize all 'regular parts'.

Unfortunately, it is unlikely that we find $H[V_h, V_i]$, $H[V_i, V_j]$, $H[V_h, V_j]$ which are all (d, ε)-regular with the same d. The so-called *Slicing Lemma* (see Lemma 6 in [11]) allows us to overcome this problem.

Suppose $H[V_h, V_i]$, $H[V_i, V_j]$, $H[V_h, V_j]$ are, resp., (d_{hi}, ε), (d_{ij}, ε), (d_{hj}, ε)-regular. Write $G^{hi} = H[V_h, V_i]$, and suppose we choose (in a careful way (see Step 2 momentarily)) numbers $\sigma_1^{hi}, \dots, \sigma_{s_{hi}}^{hi} \geq \beta$ (cf. (3)) where $\sum_{a=1}^{s_{hi}} \sigma_a^{hi} \leq d_{hi}$. The Slicing Lemma constructs, in time $O(m^2)$, a partition $G^{hi} = G_0^{hi} \cup G_1^{hi} \cup \cdots \cup G_{s_{hi}}^{hi}$, where each G_a^{hi}, $1 \leq a \leq s_{hi}$, is $(\sigma_a^{hi}, \varepsilon')$-regular (cf. (3)). Now, if there are slices $G_a^{hi} \cup G_b^{ij} \cup G_c^{hj}$ with

$$\sigma_a^{hi} = \sigma_b^{ij} = \sigma_c^{hj} \stackrel{\text{def}}{=} \sigma_{abc}^{hij}, \tag{6}$$

then the Packing Lemma builds a triangle packing $\mathscr{T}_{abc}^{hij}$ of $G_a^{hi} \cup G_b^{ij} \cup G_c^{hj}$ of size

$$\left| \mathscr{T}_{abc}^{hij} \right| \stackrel{(5)}{\geq} \left(1 - \sqrt{\zeta''}\right) \sigma_{abc}^{hij} m^2. \tag{7}$$

Now, to choose the numbers $\sigma_1^{hi}, \dots, \sigma_{s_{hi}}^{hi} \geq \beta$ above, we appeal to a so-called *Bounding Lemma* (see Lemma 7 in [11]), which returns us to Step 2 of the algorithm.

Step 2: Applying the Bounding Lemma. The Bounding Lemma is a tool which constructs, in time depending on $t \leq T_0 = O(1)$, a fractional K_3-packing ψ_0 of H_0^ω (recall H_0^ω and H_0 from Step 1) satisfying (cf. (3))

$$|\psi_0| \geq \nu_{K_3}^*(H_0^\omega) - \zeta' t^2, \quad \text{and which is } \beta\text{-}\textit{bounded}\text{, meaning that for each } \{h,i,j\} \in \tbinom{H_0}{K_3},$$
$$\psi_0(\{h,i,j\}) \geq \beta, \quad \text{or else,} \quad \psi_0(\{h,i,j\}) = 0. \tag{8}$$

Set

$$\binom{H_0}{K_3}^+ = \left\{ \{h,i,j\} \in \binom{H_0}{K_3} : \psi_0(\{h,i,j\}) \geq \beta \right\}. \tag{9}$$

The function ψ_0 defines the numbers $\sigma_j^{hi} \geq \beta$, as follows. For $\{h,i\} \in H_0$ and $\{h,i,j\} \in \tbinom{H_0}{K_3}^+$, set

$$\sigma_j^{hi} = \psi_0(\{h,i,j\}) \stackrel{(9)}{\geq} \beta. \tag{10}$$

We now apply the Slicing Lemma.

Step 3: Applying the Slicing Lemma. Fix $\{h,i\} \in H_0$. With $\left\{\sigma_j^{hi} : \{h,i,j\} \in \tbinom{H_0}{K_3}^+\right\}$ from Step 2, apply the Slicing Lemma to $G^{hi} = H[V_h, V_i]$ to construct, in time $O(n^2)$, a partition $H[V_h, V_i] = G^{hi} = G_0^{hi} \cup \bigcup \left\{ G_j^{hi} : \{h,i,j\} \in \tbinom{H_0}{K_3}^+ \right\}$ satisfying that, for each $\{h,i,j\} \in \tbinom{H_0}{K_3}^+$, G_j^{hi} is $\left(\sigma_j^{hi}, \varepsilon'\right)$-regular. Repeat over all at most $\binom{t}{2} \leq T_0^2 = O(1)$ many $\{h,i\} \in H_0$ so that Step 3 runs in time $O(n^2)$.

Step 4: Applying the Packing Lemma. Fix $\{h,i,j\} \in \tbinom{H_0}{K_3}^+$. Apply the Packing Lemma to the slices $G_j^{hi} \cup G_h^{ij} \cup G_i^{hj}$, noting that $\sigma_j^{hi} = \sigma_h^{ij} = \sigma_i^{hj} = \psi_0(\{h,i,j\}) \geq \beta$ (cf. (10)). The Packing Lemma constructs, in time polynomial in m, a triangle-packing $\mathscr{T}^{hij} = \mathscr{T}_{jhi}^{hij}$ of $G_j^{hi} \cup G_h^{ij} \cup G_i^{hj}$ of size

$$\left| \mathscr{T}^{hij} \right| \stackrel{(7)}{\geq} \left(1 - \sqrt{\zeta''}\right) \psi_0(\{h,i,j\}) m^2. \tag{11}$$

Repeat over all at most $\binom{t}{3} \leq T_0^3 = O(1)$ many $\{h,i,j\} \in \tbinom{H_0}{K_3}^+$ in time polynomial in n.

Output. Construct the family $\mathscr{T} = \bigcup \left\{ \mathscr{T}^{hij} : \{h,i,j\} \in \tbinom{H_0}{K_3}^+ \right\}$ in time $O(n^2)$. (That is, collect $O(m^2)$ triangles over at most $\binom{t}{3} \leq T_0^3 = O(1)$ indices.)

Clearly the algorithm above runs in time polynomial in n. To see that $\mathscr{T}$ is a triangle-packing of H, let $\{x,y,z\}, \{x,y,z'\} \in \mathscr{T}$. Then there exist, w.l.o.g., $1 \leq h < i < j, j' \leq t$ so that $\{x,y,z\} \in \mathscr{T}^{hij}$

and $\{x,y,z'\} \in \mathscr{T}^{hij'}$. Then $\{x,y\} \in G^{hi}_j \cap G^{hi}_{j'}$, which implies $j = j'$ since G^{hi}_j and $G^{hi}_{j'}$ are classes of a partition. Then $\{x,y,z\}, \{x,y,z'\} \in \mathscr{T}^{hij}$, which implies $z = z'$ since $\mathscr{T}^{hij}$ is a family of pairwise edge-disjoint triangles. Finally,

$$|\mathscr{T}| = \sum \left\{ \left|\mathscr{T}^{hij}\right| : \{h,i,j\} \in \binom{H_0}{K_3}^+ \right\} \overset{(11)}{\geq} \left(1 - \sqrt{\zeta''}\right) m^2 \sum \left\{ \psi_0(\{h,i,j\}) : \{h,i,j\} \in \binom{H_0}{K_3}^+ \right\}$$
$$= \left(1 - \sqrt{\zeta''}\right) m^2 \, |\psi_0| \overset{(8)}{\geq} \left(1 - \sqrt{\zeta''}\right) m^2 \left(\nu^*_{K_3}(H_0^\omega) - \zeta' t^2\right) \overset{(4)}{\geq} \left(1 - \sqrt{\zeta''}\right) \left(\nu^*_{K_3}(H) - 2\zeta' t^2 m^2\right)$$

which by (3) is at least $\nu^*_{K_3}(H) - 4\zeta' n^2 = \nu^*_{K_3}(H) - \zeta n^2$. □

1.3. **Itinerary of paper.** To prove Theorem 1.3, we shall follow the same approach outlined above for the graph case. As such, we need 3-uniform hypergraph analogues of each of the tools sketched in the previous section. We proceed along the following itinerary. In Section 2, we present algorithmic (3-uniform) tools of the following forms:

- a *Regularity Lemma* (upcoming Theorem 2.12) due to Haxell, Nagle and Rödl [9].
- a *Packing Lemma* (upcoming Lemma 2.7), which we prove in Sections 4–5;
- a *Slicing Lemma* (upcoming Lemma 2.4), which we prove in Section 6;
- a *Bounding Lemma* (upcoming Lemma 2.18), taken from [11, 10].

In Section 3, we use these tools to prove our main result, Theorem 1.3.

Remark 1.6. The most important tools in this paper are the Regularity Lemma and the Packing Lemma. In essence, the Packing Lemma is a consequence of a so-called Counting Lemma from [9] (see Theorem 5.2 in this paper). Since the Regularity Lemma and the Counting Lemma were developed in [9], Theorem 1.3 seemed possible if one followed the approach of Haxell and Rödl [11] outlined above.

In the hypergraph setting, deriving the Packing Lemma from the Counting Lemma is somewhat technical, despite following standard lines. We derive the Packing Lemma from a so-called *Extension Lemma* (see Lemma 4.3 in this paper), which we in turn derive from the Counting Lemma. These tools could be of potential use in other settings.

The algorithmic aspects of the Slicing Lemma are of a less standard nature. The Slicing Lemma could be of use in other contexts, and it may be of independent interest. □

1.4. **A minor technicality.** In our outline, we took $F_0 = K_3$ to be the triangle, which illustrates all but one detail in the work of Haxell and Rödl [11]. In particular, whenever F_0 is not complete, one also needs a so-called *Crossing Lemma* from [11] (see Lemma 4 there), which we now state for 3-graphs. For 3-graphs $\mathcal{F}_0$ and $\mathcal{H}$, and for a partition $\Pi_0 : V(\mathcal{H}) = U_1 \cup \cdots \cup U_k$, we say that $\mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}$ *crosses* Π_0 if $|V(\mathcal{F}) \cap U_i| \leq 1$ for all $1 \leq i \leq k$. We write $\binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0}$ for the family of all crossing copies of $\mathcal{F}$ in $\mathcal{H}$.

Lemma 1.7 (Crossing Lemma [11]). *For every* 3*-graph* $\mathcal{F}_0$ *on* f *vertices, and for all* $\xi > 0$*, there exists* $K_0 = K_0(\xi, \mathcal{F}_0)$ *so that the following holds.*

Let $\mathcal{H}$ *be a* 3*-graph on* n *vertices, and let* ψ *be a fractional* $\mathcal{F}_0$*-packing of* $\mathcal{H}$*. There exists an algorithm which constructs, in time* $O(n^f)$*, a vertex partition* $\Pi_0 : V(\mathcal{H}) = U_1 \cup \cdots \cup U_k$*, for some* $k \leq K_0$*, where* $|U_1| \leq \cdots \leq |U_k| \leq |U_1| + 1$*, satisfying that* $|\psi_{\Pi_0}| \overset{\text{def}}{=} \sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0} \right\} \geq (1 - \xi)|\psi|$.

Haxell and Rödl [11] proved Lemma 1.7 (see Lemma 11 there) in a setting more general than that of (k-uniform) hypergraphs and fractional packings. (See Remark 2.10 in [4] for some related comments.)

In our previous outline, if F_0 were not complete, the Crossing Lemma would appear as Step 0. We would construct a fractional F_0-packing ψ of H of size $|\psi| = \nu^*_{F_0}(H)$ (via linear programming, running in time polynomial in n). We would then apply the Crossing Lemma to H and ψ to construct Π_0 in time $O(n^f)$. In Step 1, we would require Π to *refine* Π_0 in the usual way (which is always possible with a regularity lemma). All remaining details would proceed as we described.

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2. Algorithmic 3-graph Regularity Tools

In this section, we present (1) the Slicing Lemma (Lemma 2.4), (2) the Packing Lemma (Lemma 2.7), (3) the Regularity Lemma (Theorem 2.12), and (4) the Bounding Lemma (Lemma 2.18), where this order is determined by inclusion of the concepts needed to present each statement.

2.1. (α, δ)-minimality.

For graphs, ε-*regularity* may be viewed as the central concept of the Introduction. For 3-graphs, we shall consider the corresponding concept of (α, δ)-*minimality* (see upcoming Definition 2.3). For that concept, triples of a 3-graph $\mathcal{G}$ will be defined on pairs from an underlying graph P. To make this precise, for a graph P, write $\mathcal{K}_3(P) = \{\{v_1, v_2, v_3\} : \{v_i, v_j\} \in P \text{ for all } 1 \leq i < j \leq 3\}$ for the family of all *triangles* K_3 of P. We then say P *underlies* $\mathcal{G}$ if $\mathcal{G} \subseteq \mathcal{K}_3(P)$. Whenever $P' \subseteq P$ satisfies $\mathcal{K}_3(P') \neq \emptyset$, we define the *density* $d_{\mathcal{G}}(P')$ *of* $\mathcal{G}$ *w.r.t.* P' by $d_{\mathcal{G}}(P') = |\mathcal{G} \cap \mathcal{K}_3(P')|/|\mathcal{K}_3(P')|$. Unless otherwise indicated, we reserve the symbol α for $\alpha = d_{\mathcal{G}}(P) = |\mathcal{G}|/|\mathcal{K}_3(P)|$.

In the context above, the underlying graphs P will be 3-partite, balanced, and well-behaved. We call the following environment a *triad*.

Definition 2.1 (triad)**.** Let $d \geq 0$, $\varepsilon > 0$, and $m \in \mathbb{N}$ be given. We call a graph P a *triad* if $P = P^{12} \cup P^{23} \cup P^{13}$ is 3-partite with vertex partition $V(P) = V_1 \cup V_2 \cup V_3$, where $|V_1| = |V_2| = |V_3| = m$, and where each $P^{ij} = P[V_i, V_j]$ is (d, ε)-regular, $1 \leq i < j \leq 3$.

Throughout this paper, we use the following well-known fact (cf. upcoming Lemma 7.3).

Fact 2.2 (triangle counting lemma)**.** *For all $d, \tau > 0$, there exists $\varepsilon = \varepsilon_{\text{Fact 2.2}}(d, \tau) > 0$ so that whenever P is a triad with the parameters $d, \varepsilon > 0$ and m sufficiently large, then $|\mathcal{K}_3(P)| = (1 \pm \tau) d^3 m^3$.*

Now, let P be a triad and let $\mathcal{G} \subseteq \mathcal{K}_3(P)$ be given with $\alpha = d_{\mathcal{G}}(P)$. As usual, let $K^{(3)}_{2,2,2}$ denote the complete 3-partite 3-graph with 2 vertices in each vertex class. We then define the family $\mathcal{K}_{2,2,2}(\mathcal{G}) = \left\{ J \in \binom{V(\mathcal{G})}{6} : J \text{ induces a copy of } K^{(3)}_{2,2,2} \text{ in } \mathcal{G} \right\}$. In the context of Definition 2.1, if $\varepsilon = \varepsilon(\alpha, d) > 0$ is sufficiently small and $m = m(\alpha, d, \varepsilon)$ is sufficiently large, it is not difficult to prove (see [9] for a proof) that $|\mathcal{K}_{2,2,2}(\mathcal{G})| \geq \alpha^8 d^{12} \binom{m}{2}^3 \left(1 - \varepsilon^{1/10}\right)$. The following concept is therefore motivated.

Definition 2.3 ((α, δ)-minimality)**.** Let P and $\mathcal{G} \subseteq \mathcal{K}_3(P)$ be given as in Definition 2.1 with $\alpha = d_{\mathcal{G}}(P)$. For $\delta > 0$, we say $\mathcal{G}$ is (α, δ)-*minimal w.r.t.* P if $|\mathcal{K}_{2,2,2}(\mathcal{G})| \leq \alpha^8 d^{12} \binom{m}{2}^3 (1 + \delta)$.

2.2. The Slicing Lemma.

With the definitions above, we can already present the Slicing Lemma. (In what follows, $x = y \pm z$ denotes $y - z \leq x \leq y + z$.)

Lemma 2.4 (Slicing Lemma)**.** *For all $\alpha_0, \delta' > 0$, there exists $\delta = \delta_{\text{Lem.2.4}}(\alpha_0, \delta') > 0$ so that, for all $d > 0$, there exists $\varepsilon = \varepsilon_{\text{Lem.2.4}}(\alpha_0, \delta', \delta, d) > 0$ so that the following statement holds.*

Let P be a triad with parameters d, ε and a sufficiently large integer m. Let $\mathcal{G} \subseteq \mathcal{K}_3(P)$ be (α, δ)-minimal w.r.t. P, for some $\alpha \geq \alpha_0$. Suppose $\sigma_1, \ldots, \sigma_s \geq \alpha_0$ are given with $\sum_{i=1}^{s} \sigma_i \leq \alpha$. Then, in time $O(m^3)$, one can construct a partition $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_s$ so that, for each $1 \leq i \leq s$, $\mathcal{G}_i$ is (α_i, δ')-minimal w.r.t. P, where $\alpha_i = d_{\mathcal{G}_i}(P) = \sigma_i \pm \delta'$.

We prove Lemma 2.4 in Section 6.

2.3. **The Packing Lemma.** To present the Packing Lemma (Lemma 2.7), we require some additional considerations. We summarize these conditions in the following environment and subsequent definition.

Setup 2.5 (Packing Setup)**.** Suppose $\alpha, \delta, d, \varepsilon > 0$ and $f, m \in \mathbb{N}$ are given together with a fixed 3-graph $\mathcal{F}_0$ on the vertex set $[f] = \{1, \dots, f\}$. (This will be the same $\mathcal{F}_0$ appearing in Theorem 1.3.) Suppose

(1) $P = \bigcup\{P^{ij} : 1 \le i < j \le f\}$ is an f-partite graph with vertex partition $V_1 \cup \dots \cup V_f$, where $|V_1| = \dots = |V_f| = m$, and where each $P^{ij} = P[V_i, V_j]$ is (d, ε)-regular, $1 \le i < j \le f$;

(2) $\mathcal{G} = \bigcup\{\mathcal{G}^{hij} : 1 \le h < i < j \le f\} \subseteq \mathcal{K}_3(P)$ is a 3-graph satisfying that, for each $1 \le h < i < j \le f$, $\mathcal{G}^{hij} = \mathcal{G}[V_h, V_i, V_j]$ is (α_{hij}, δ)-minimal w.r.t. $P^{hi} \cup P^{ij} \cup P^{hj}$, where $\alpha_{hij} = \alpha \pm \delta$ if $\{h, i, j\} \in \mathcal{F}_0$, and $\alpha_{hij} = 0$ otherwise (i.e., $\mathcal{G}^{hij} = \emptyset$ otherwise).

Definition 2.6 (partite-isomorphic)**.** Let $\mathcal{F}_0$, P and $\mathcal{G}$ be given as in Setup 2.5. Let $\mathcal{F} \subset \mathcal{G}$ be a subhypergraph of $\mathcal{G}$ on vertices $v_1, \dots, v_f$, where $v_1 \in V_1, \dots, v_f \in V_f$. We say that $\mathcal{F}$ is a *partite-isomorphic copy* of $\mathcal{F}_0$ if for every $1 \le i < j \le f$, $\{v_i, v_j\} \in P^{ij}$, and if $v_i \mapsto i$ defines an isomorphism from $\mathcal{F}$ to $\mathcal{F}_0$.

Lemma 2.7 (Packing Lemma)**.** *Let $\mathcal{F}_0$ be a fixed 3-graph with $V(\mathcal{F}_0) = [f]$. For all $\alpha_0, \rho > 0$, there exists $\delta = \delta_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho) > 0$ so that, for all $d^{-1} \in \mathbb{N}$, there exists $\varepsilon = \varepsilon_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho, \delta, d) > 0$ so that the following holds.*

Let P and $\mathcal{G}$ satisfy the hypothesis of Setup 2.5 with $\mathcal{F}_0$ and some $\alpha \ge \alpha_0$, with $\delta, d, \varepsilon > 0$ above, and with a sufficiently large integer m. Then, one may construct, in time polynomial in m, an $\mathcal{F}_0$-packing $\mathscr{F}_{\mathcal{G}}$ of $\mathcal{G}$ covering all but $\rho|\mathcal{G}|$ edges of $\mathcal{G}$, where $\mathscr{F}_{\mathcal{G}}$ consists entirely of partite-isomorphic copies of $\mathcal{F}_0$ in $\mathcal{G}$. One has, in particular, $|\mathscr{F}_{\mathcal{G}}| \ge (1 - 2\rho)\alpha d^3 m^3$.

We prove Lemma 2.7 in Section 4.

Remark 2.8. Lemma 2.7 holds whether $d > 0$ satisfies $d^{-1} \in \mathbb{N}$ or not. We make the assumption here because results we use from [9] to prove Lemma 2.7 made the same assumption. (Moreover, it suffices to take $d^{-1} \in \mathbb{N}$ because the Regularity Lemma always provides this condition.) □

Remark 2.9. The last assertion of Lemma 2.7 is an easy consequence of its predecessor. To see this, we may assume, w.l.o.g., that $2\delta \le \rho \times \alpha_0^2$ and that, with $\tau = \delta$, we have $\varepsilon \le \varepsilon_{\text{Fact 2.2}}(d, \tau)$. Now, let $\mathcal{G}' \subseteq \mathcal{G}$ denote the set of edges covered by $\mathscr{F}_{\mathcal{G}}$. Then every element $\mathcal{F} \in \mathscr{F}_{\mathcal{G}}$ covers precisely $|\mathcal{F}_0|$ edges of $\mathcal{G}'$, and every edge of $\mathcal{G}'$ is covered by precisely one element $\mathcal{F} \in \mathscr{F}_{\mathcal{G}}$. Thus,

$$\begin{aligned} |\mathscr{F}_{\mathcal{G}}| \times |\mathcal{F}_0| = |\mathcal{G}'| \ge (1-\rho)|\mathcal{G}| &= (1-\rho) \sum \left\{ |\mathcal{G}^{hij}| : \{h, i, k\} \in \mathcal{F}_0 \right\} \\ &= (1-\rho) \sum \left\{ \alpha_{hij} |\mathcal{K}_3(P^{hi} \cup P^{ij} \cup P^{hj})| : \{h, i, j\} \in \mathcal{F}_0 \right\} \\ &\ge (1-\rho)(\alpha - \delta) \sum \left\{ |\mathcal{K}_3(P^{hi} \cup P^{ij} \cup P^{hj})| : \{h, i, j\} \in \mathcal{F}_0 \right\} \\ &\overset{\text{Fact 2.2}}{\ge} |\mathcal{F}_0| \times (1-\rho)(\alpha-\delta)(1-\delta) d^3 m^3 \ge |\mathcal{F}_0| \times (1-\rho) \left(1 - \frac{\delta}{\alpha_0}\right)(1-\delta)\alpha d^3 m^3, \end{aligned}$$

and so now the result follows by a few routine calculations. □

2.4. **The Regularity Lemma.** We now present the Regularity Lemma from [9] (see Theorem 2.5 there). For a 3-graph $\mathcal{H}$, the Regularity Lemma from [9] will partition the vertices $V = V(\mathcal{H})$ and partition the pairs $\binom{V}{2}$, in such a way that the following holds.

Definition 2.10 ((ℓ, t, ε)-partition)**.** Let $\ell, t \in \mathbb{N}$ and $\varepsilon > 0$ be given, and suppose V is a set with size $|V| = n$. An *(equitable) (ℓ, t, ε)-partition* of V is a pair $(\Pi, \mathscr{P})$ of partitions of the following form:

(1) $\Pi : V = V_0 \cup V_1 \cup \dots \cup V_t$ is a t-equitable partition of V, i.e, $|V_1| = \dots = |V_t| \overset{\text{def}}{=} m \ge \lfloor n/t \rfloor$;

(2) $\mathscr{P}$ is a partition of $K[V_1, \dots, V_t]$ with classes, for each $1 \le i < j \le t$, $K[V_i, V_j] = P_1^{ij} \cup \dots \cup P_\ell^{ij}$, where every $P_a^{ij} \in \mathscr{P}$ is (ℓ^{-1}, ε)-regular.

For a 3-graph $\mathcal{H}$, the Regularity Lemma from [9] will construct an (ℓ, t, ε)-partition of $V(\mathcal{H})$ and a 'large' subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$ with the following property.

Definition 2.11 ((α_0,δ)-minimal partition)**.** Let $\ell, t \in \mathbb{N}$ and $\alpha_0, \delta, \varepsilon > 0$ be given, and suppose $\mathcal{H}$ is a 3-graph and $(\Pi, \mathscr{P})$ is an (ℓ, t, ε)-partition of $V(\mathcal{H})$. For a subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$, we say $(\Pi, \mathscr{P})$ is (α_0, δ)-*minimal w.r.t.* $\mathcal{H}'$ if for every $\{x, y, z\} \in \mathcal{H}'$, there exist $1 \le h < i < j \le t$ and $1 \le a, b, c \le \ell$ so that $\{x, y, z\} \in \mathcal{K}_3(P_a^{hi} \cup P_b^{ij} \cup P_c^{hj})$, where $\mathcal{H} \cap \mathcal{K}_3(P_a^{hi} \cup P_b^{ij} \cup P_c^{hj})$ is $(\alpha_{abc}^{hij}, \delta)$-minimal w.r.t. $P_a^{hi} \cup P_b^{ij} \cup P_c^{hj}$ for some $\alpha_{abc}^{hij} \ge \alpha_0$.

We now give the regularity lemma from [9] (see Theorem 2.5 there).

Theorem 2.12 (Regularity Lemma [9])**.** *For all $\alpha_0, \delta > 0$, and for all functions $\boldsymbol{\varepsilon} : \mathbb{N} \to (0,1)$, there exist positive integers $T_0 = T_0(\alpha_0, \delta, \boldsymbol{\varepsilon})$, $L_0 = L_0(\alpha_0, \delta, \boldsymbol{\varepsilon})$, and $N_0 = N_0(\alpha_0, \delta, \boldsymbol{\varepsilon})$ so that the following holds.*

Let $\mathcal{H}$ be a 3-graph with vertex set $V = V(\mathcal{H})$, where $|V| = n > N_0$. Then, in time $O(n^6)$, one can construct an $(\ell, t, \boldsymbol{\varepsilon}(\ell))$-partition $(\Pi, \mathscr{P})$ of V, for some $\ell \le L_0$ and some $t \le T_0$, and a subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$, where $|\mathcal{H}'| > |\mathcal{H}| - (\alpha_0 + \delta)n^3 - tn^2$, and with respect to which $(\Pi, \mathscr{P})$ is (α_0, δ)-minimal.

We make the following Remark for future reference.

Remark 2.13. In Theorem 2.12, suppose a fixed integer $k \ge 1$ is given with α_0, δ, and $\boldsymbol{\varepsilon} : \mathbb{N} \to (0,1)$. (The constants T_0, L_0 and N_0 will now depend also on k.) Suppose $\mathcal{H}$ is given with a pre-partition $\Pi_0 : V = U_1 \cup \cdots \cup U_k$, where $|U_1| \le \cdots \le |U_k| \le |U_1| + 1$. If we allow $|V_0|$ to be as large as $t + k$, i.e.,

$$|\mathcal{H}'| > |\mathcal{H}| - (\alpha_0 + \delta)n^3 - (t + k)n^2, \tag{12}$$

then the proof of Theorem 2.12 allows that Π can be taken to refine Π_0, in the sense that for each $1 \le i \le t$, there exists $1 \le i' \le k$ so that $V_i \subseteq U_{i'}$. In this context, we shall call a triple $1 \le h < i < j \le t$ *transversal* if $V_h \subseteq U_{h'}$, $V_i \subseteq U_{i'}$, and $V_j \subseteq U_{j'}$, where h', i', j' are distinct. Since the integers $t \le T_0$ and $k \le K_0$ (where K_0 will be given by Lemma 1.7) will always be constants in this paper, while $n \to \infty$ whenever needed, we abbreviate (12) to say

$$|\mathcal{H}'| > |\mathcal{H}| - (\alpha_0 + \delta)n^3 - O(n^2). \tag{13}$$

□

2.5. **The Bounding Lemma.** We now present the Bounding Lemma from [10], which appeared as Lemma 3.6 there. The Bounding Lemma concerns fractional packings in weighted multi-hypergraphs related to (ℓ, t, ε)-partitions (cf. Definition 2.10). To make this precise, we require several definitions, taken mostly from [10].

Definition 2.14 ((ℓ,t)-augmented (weighted) 3-graph)**.** On vertex set $[t] = \{1, 2, \ldots, t\}$, let $M = M^{(2)}(\ell, t) = \left\{p_a^{ij} : 1 \le i < j \le t,\, 1 \le a \le \ell\right\}$ be the complete multigraph with edge-multiplicity ℓ, where the set of multiedges connecting $1 \le i < j \le t$ is $\{p_1^{ij}, \ldots, p_\ell^{ij}\}$. We call M the *complete (ℓ,t)-multigraph*. Define $\mathcal{M} = \mathcal{M}^{(3)}(\ell, t) = \left\{\left\{p_a^{hi}, p_b^{ij}, p_c^{hi}\right\} : 1 \le h < i < j \le t,\, 1 \le a, b, c \le \ell\right\}$ to be the *complete (ℓ,t)-augmented 3-graph.* Any subset $\mathcal{A} \subseteq \mathcal{M}$ is called an *(ℓ,t)-augmented 3-graph.* If $\omega : \mathcal{M} \to [0,1]$ is a weight function, then $\mathcal{A} = \omega^{-1}(0,1]$ is an (ℓ,t)-augmented 3-graph, and we define $\mathcal{A}^\omega = \{(A, \omega(A)) : A \in \mathcal{A}\}$ to be the *(ℓ,t)-augmented ω-weighted 3-graph.*

Clearly, (ℓ,t)-augmented and ω-weighted 3-graphs $\mathcal{A}^\omega$ provide the 'cluster objects' of the Regularity Lemma (Theorem 2.12). We make this precise in the following remark.

Remark 2.15. For a set V, an (ℓ, t, ε)-partition $(\Pi, \mathscr{P})$ of V corresponds to the complete (ℓ,t)-multigraph M defined above, where $P_a^{hi} \in \mathscr{P}$ corresponds to $p_a^{hi} \in M$. The family of all triads of $(\Pi, \mathscr{P})$ corresponds to the complete (ℓ,t)-augmented 3-graph $\mathcal{M}$ defined above, where

$$A = \left\{p_a^{hi}, p_b^{ij}, p_c^{hi}\right\} \in \mathcal{M} \quad \text{corresponds to the triad} \quad P^A \stackrel{\text{def}}{=} P_a^{hi} \cup P_b^{ij} \cup P_c^{hj} \subset \mathscr{P}. \tag{14}$$

If $V = V(\mathcal{H})$ and $(\Pi, \mathscr{P})$ is (α_0, δ)-minimal w.r.t. $\mathcal{H}' \subseteq \mathcal{H}$, then $\mathcal{H}'$ and $(\Pi, \mathscr{P})$ will correspond to an (ℓ,t)-augmented 3-graph $\mathcal{A}$. Indeed, for $A \in \mathcal{M}$, write

$$\mathcal{H}^A \stackrel{\text{def}}{=} \mathcal{H} \cap \mathcal{K}_3\left(P^A\right) \quad \text{and} \quad \alpha^A \stackrel{\text{def}}{=} d_{\mathcal{H}}\left(P^A\right), \tag{15}$$

and so one would take

$$A \in \mathcal{A} \quad \Longleftrightarrow \quad \mathcal{H}^A \text{ is } (\alpha^A, \delta)\text{-minimal w.r.t. } P^A \text{ for some } \alpha^A \geq \alpha_0. \tag{16}$$

More generally, we can define weight function $\omega : \mathcal{M} \to [0,1]$ by

$$\omega(A) = \begin{cases} \alpha^A & \text{if } \alpha^A \geq \alpha_0 \text{ and } \mathcal{H}^A \text{ is } (\alpha^A, \delta)\text{-minimal w.r.t. } P^A, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Then $\mathcal{A} = \omega^{-1}(0,1]$ is the (ℓ,t)-augmented 3-graph defined in (16), and $\mathcal{A}^\omega = \{(A, \omega(A)) : A \in \mathcal{A}\}$ is an (ℓ,t)-augmented ω-weighted 3-graph. When Π has refined a pre-partition Π_0 of V (cf. Remark 2.13), we alter (17) to say

$$\omega(A) = \begin{cases} \alpha^A & \text{if } A \text{ is transversal, } \alpha^A \geq \alpha_0, \text{ and } \mathcal{H}^A \text{ is } (\alpha^A, \delta)\text{-minimal w.r.t. } P^A, \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

and define $\mathcal{A}$ and $\mathcal{A}^\omega$ identically to before. □

For an (ℓ,t)-augmented 3-graph $\mathcal{A}$, we next define *copies* $\mathcal{F} \subset \mathcal{A}$ of a fixed 3-graph $\mathcal{F}_0$ in $\mathcal{A}$, and copies *containing* a fixed edge $A \in \mathcal{A}$.

Definition 2.16 (copy, edge containment)**.** Let $\mathcal{F}_0$ be a 3-graph and let $\mathcal{A}$ be an (ℓ,t)-augmented 3-graph. A *copy* $\mathcal{F}$ of $\mathcal{F}_0$ in $\mathcal{A}$ is a pair (ϕ_1, ϕ_2) of functions, where the first function $\phi_1 : V(\mathcal{F}_0) \to [t]$ is an injection and where we write $\Phi_1 \stackrel{\text{def}}{=} \phi_1(V(\mathcal{F}_0))$, and where the second function $\phi_2 : \binom{\Phi_1}{2} \to [\ell]$ satisfies

$$\{u,v,w\} \in \mathcal{F}_0 \quad \Longrightarrow \quad \left\{p_a^{hi}, p_b^{ij}, p_c^{hj}\right\} \in \mathcal{A} \quad \text{where}$$
$$\{h,i,j\} = \phi_1(\{u,v,w\}) \quad \text{and} \quad (a,b,c) = (\phi_2(\{h,i\}), \phi_2(\{i,j\}), \phi_2(\{h,j\})).$$

In reverse, suppose $A = \{p_a^{hi}, p_b^{ij}, p_c^{hj}\} \in \mathcal{A}$ and $\mathcal{F} = (\phi_1, \phi_2)$ is a copy of $\mathcal{F}_0$ in $\mathcal{A}$. We say $\mathcal{F}$ *contains* A, and write $A \in \mathcal{F}$, if $\phi_1^{-1}(\{h,i,j\}) \in \mathcal{F}_0$ and if $(a,b,c) = (\phi_2(\{h,i\}), \phi_2(\{i,j\}), \phi_2(\{h,j\})$. Finally, we write $\binom{\mathcal{A}}{\mathcal{F}_0}$ to denote the family of all copies of $\mathcal{F}_0$ in $\mathcal{A}$, and $\binom{\mathcal{A}}{\mathcal{F}_0}_A = \left\{\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0} : A \in \mathcal{F}\right\}$.

In what follows, we define a fractional $\mathcal{F}_0$-packing of an (ℓ,t)-augmented weighted 3-graph $\mathcal{A}^\omega$ (the definition is identical to (2)), and we also define a concept of boundedness.

Definition 2.17 ((β-bounded) fractional $\mathcal{F}_0$-packing of $\mathcal{A}^\omega$)**.** Let $\mathcal{A}^\omega$ be an (ℓ,t)-augmented ω-weighted 3-graph. A *fractional* $\mathcal{F}_0$*-packing* of $\mathcal{A}^\omega$ is a function $\psi : \binom{\mathcal{A}}{\mathcal{F}_0} \to [0,1]$ satisfying that, for each $A \in \mathcal{A}$, $\sum\left\{\psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A\right\} \leq \omega(A)$. For a fractional $\mathcal{F}_0$-packing ψ of $\mathcal{A}^\omega$, we write $|\psi| = \sum\left\{\psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}\right\}$ for the *size* of ψ, and we write $\nu^*_{\mathcal{F}_0}(\mathcal{A})$ for the maximum size of a fractional $\mathcal{F}_0$-packing of $\mathcal{A}$. For $\beta > 0$, we say that ψ is β*-bounded* if for every $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}$, $\psi(\mathcal{F}) \geq \beta$, or else, $\psi(\mathcal{F}) = 0$.

We may finally state the Bounding Lemma from [10] (see Lemma 3.6 there).

Lemma 2.18 (Bounding Lemma)**.** *For every 3-graph $\mathcal{F}_0$ and for all positive η, there exists $\beta = \beta_{\text{Lem.2.18}}(\mathcal{F}_0, \eta) > 0$ so that, for every (ℓ,t)-augmented ω-weighted 3-graph $\mathcal{A}^\omega$, there exists a β-bounded fractional $\mathcal{F}_0$-packing ψ of $\mathcal{A}^\omega$ so that $|\psi| \geq \nu^*_{\mathcal{F}_0}(\mathcal{A}^\omega) - \eta\ell^3t^3$. Moreover, ψ may be constructed in time depending on ℓ and t.*

Remark 2.19. Lemma 2.18 was proven in [10], but without regard to the constructive assertion. However, this assertion follows easily from the proof in [10], which consists of an application of a statement of Haxell and Rödl appearing as Theorem 18 in [11]. Theorem 18, in turn, is proven by a standard probabilistic argument on a family of sets on $\ell^3t^3 = O(1)$ vertices, which one exhaustively derandomizes. Algorithmic aspects are briefly discussed by Haxell and Rödl in [11] (see Section 3). □

3. Proof of Main Result

In this section, we prove Theorem 1.3 in Steps 0–4. In particular, Step 0 will apply the Crossing Lemma (Lemma 1.7), and Steps 1–4 will align with those in the Introduction. We begin by discussing our input, and by defining some auxiliary constants and parameters. The Reader not interested in these details may refer to the hierarchies provided below in (22), (26) and (28).

Input and auxiliary constants. Let $\mathcal{F}_0$ be a fixed 3-graph on f vertices, and let $\zeta > 0$ be given. Set

$$\xi = \rho = \eta = \tau = \frac{\zeta}{8}. \tag{19}$$

We now define some constants related to the Crossing and Bounding Lemmas. With ξ given above, let

$$K_0 = K_0(\xi, \mathcal{F}_0) \tag{20}$$

be the constant guaranteed by the Crossing Lemma (Lemma 1.7). With $\eta > 0$ given above, let

$$\beta = \beta_{\text{Lem.2.18}}(\mathcal{F}_0, \eta) > 0 \tag{21}$$

be the constant guaranteed by the Bounding Lemma (Lemma 2.18). We have the first hierarchy

$$\zeta > \xi = \rho = \eta = \tau \gg \frac{1}{K_0}, \beta. \tag{22}$$

We next define some constants related to the Packing and Slicing Lemmas. With β given above, set

$$\alpha_0 = \beta. \tag{23}$$

With ρ in (19) and α_0 above, let $\delta_{\text{Lem.2.7}} = \delta_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho) > 0$ be the constant guaranteed by the Packing Lemma (Lemma 2.7). With α_0 in (23) and $\delta' = \delta_{\text{Lem.2.7}}$, let $\delta_{\text{Lem.2.4}} = \delta_{\text{Lem.2.4}}(\alpha_0, \delta_{\text{Lem.2.7}}) > 0$ be the constant guaranteed by the Slicing Lemma (Lemma 2.4). It is the case that $\delta_{\text{Lem.2.4}} < \delta_{\text{Lem.2.7}}$ (as will be seen in the proof of Lemma 2.4), and so we set

$$\delta = \delta_{\text{Lem.2.4}}. \tag{24}$$

Continuing, let $\ell \in \mathbb{N}$ be an integer variable. With ρ in (19), α_0 in (23), $\delta_{\text{Lem.2.7}}$ above and $d = 1/\ell$, let $\boldsymbol{\varepsilon}_{\text{Lem.2.7}}(\ell) = \boldsymbol{\varepsilon}_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho, \delta_{\text{Lem.2.7}}, 1/\ell) > 0$ be the function (of integer variable ℓ) guaranteed by the Packing Lemma (Lemma 2.7). With α_0 in (23), $\delta' = \delta_{\text{Lem.2.7}}$ above, $\delta = \delta_{\text{Lem.2.4}}$ in (24) and $d = 1/\ell$, let $\boldsymbol{\varepsilon}_{\text{Lem.2.4}}(\ell) = \boldsymbol{\varepsilon}_{\text{Lem.2.4}}(\alpha_0, \delta_{\text{Lem.2.7}}, \delta, 1/\ell) > 0$ be the function guaranteed by the Slicing Lemma (Lemma 2.4). Finally, with $\tau > 0$ given in (19) and $d = 1/\ell$, let $\boldsymbol{\varepsilon}_{\text{Fact 2.2}}(\ell) = \boldsymbol{\varepsilon}_{\text{Fact 2.2}}(1/\ell, \tau) > 0$ be the function guaranteed by the Triangle Counting Lemma (Fact 2.2). Set

$$\boldsymbol{\varepsilon}(\ell) = \min\left\{\boldsymbol{\varepsilon}_{\text{Lem.2.7}}(\ell), \boldsymbol{\varepsilon}_{\text{Lem.2.4}}(\ell), \boldsymbol{\varepsilon}_{\text{Fact2.2}}(\ell)\right\}. \tag{25}$$

With integer variable ℓ, we have the second hierarchy (cf. (22))

$$\beta \gg \delta_{\text{Lem.2.7}} \gg \delta_{\text{Lem.2.4}} = \delta \geq \min\left\{\frac{1}{\ell}, \delta\right\} \gg \boldsymbol{\varepsilon}_{\text{Lem.2.7}}, \boldsymbol{\varepsilon}_{\text{Lem.2.4}}, \boldsymbol{\varepsilon}_{\text{Fact 2.2}}(\ell) \geq \boldsymbol{\varepsilon}(\ell). \tag{26}$$

Finally, we define some constants related to the Regularity Lemma. Let k be an integer variable. With constants α_0 in (23) and δ in (24), and with function $\boldsymbol{\varepsilon}$ in (25), let $\boldsymbol{T}_0(k) = \boldsymbol{T}_0(k, \alpha_0, \delta, \boldsymbol{\varepsilon})$, $\boldsymbol{L}_0(k) = \boldsymbol{L}_0(k, \alpha_0, \delta, \boldsymbol{\varepsilon})$, and $\boldsymbol{N}_0(k) = \boldsymbol{N}_0(k, \alpha_0, \delta, \boldsymbol{\varepsilon})$ be the functions (of integer variable k) guaranteed by Theorem 2.12 (cf. Remark 2.13). (These are not functions of ℓ.) With K_0 from (20), let

$$T_0 = \max_{1 \leq k \leq K_0} \boldsymbol{T}_0(k), \quad L_0 = \max_{1 \leq k \leq K_0} \boldsymbol{L}_0(k), \quad N_0 = \max_{1 \leq k \leq K_0} \boldsymbol{N}_0(k). \tag{27}$$

We take integer n_0 so that, in the final hierarchy (cf. (22), (26))

$$\min_{1 \leq \ell \leq L_0} \boldsymbol{\varepsilon}(\ell) \gg \frac{1}{T_0}, \frac{1}{L_0}, \frac{1}{N_0} \ggg \frac{1}{n_0}. \tag{28}$$

Now, let $\mathcal{H}$ be a given 3-graph on $n \geq n_0$ vertices. We construct, in time polynomial in n, an $\mathcal{F}_0$-packing $\mathscr{F}_{\mathcal{H}}$ of $\mathcal{H}$ of size

$$|\mathscr{F}_{\mathcal{H}}| \geq \nu^*_{\mathcal{F}_0}(\mathcal{H}) - \zeta n^3. \tag{29}$$

Since $\nu^*_{\mathcal{F}_0}(\mathcal{H}) \geq \nu_{\mathcal{F}_0}(\mathcal{H})$, this will prove Theorem 1.3. We proceed to the first step of the algorithm.

Step 0: Applying the Crossing Lemma. Our first step is to apply the Crossing Lemma to $\mathcal{H}$. (We do so in order to prove (32) below.) For that purpose, construct a maximum fractional $\mathcal{F}_0$-packing $\psi : \binom{\mathcal{H}}{\mathcal{F}_0} \to [0,1]$, i.e., one for which $|\psi| = \nu^*_{\mathcal{F}_0}(\mathcal{H})$ (which is a linear programming problem running in time polynomial in n). With $\xi > 0$ in (19), we apply the Crossing Lemma (Lemma 1.7) to $\mathcal{H}$ and ψ to construct, in time $O(n^f)$, a partition $\Pi_0 : V(\mathcal{H}) = U_1 \cup \cdots \cup U_k$, where $k \leq K_0$ (see (20)) and $|U_1| \leq \cdots \leq |U_k| \leq |U_1| + 1$, and where

$$|\psi_{\Pi_0}| = \sum \left\{ \psi(F) : F \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0} \right\} \geq (1-\xi)|\psi| = (1-\xi)\nu^*_{\mathcal{F}_0}(\mathcal{H}). \tag{30}$$

(recall the notation $\binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0}$ from Lemma 1.7).

Step 1: Applying the Regularity Lemma. Our next step is to apply the Regularity Lemma (Theorem 2.12) to $\mathcal{H}$ and its vertex partition $V(\mathcal{H}) = U_1 \cup \cdots \cup U_k$ from Step 1. To that end, recall the constants α_0 in (23) and δ in (24), the integer k above (where $k \leq K_0$ (cf. (20))), and the function $\boldsymbol{\varepsilon}$ in (25). With these parameters, Theorem 2.12 constructs, in time $O(n^6)$, an $(\ell, t, \boldsymbol{\varepsilon}(\ell))$-partition $(\Pi, \mathscr{P})$ of $V(\mathcal{H})$, for some $\ell \leq L_0$ and some $t \leq T_0$ (cf. (27)), which refines Π_0 (cf. Remark 2.13), and constructs a subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$, where $|\mathcal{H}'| > |\mathcal{H}| - (\alpha_0 + \delta)n^3 - O(n^2)$, and with respect to which $(\Pi, \mathscr{P})$ is (α_0, δ)-minimal. To simplify notation slightly, now that Theorem 2.12 has been applied, the integers $\ell \leq L_0$ and $t \leq T_0$ are *fixed* (they are no longer variables), and so we shall write (cf. (25))

$$\varepsilon = \boldsymbol{\varepsilon}(\ell), \quad \varepsilon_{\text{Lem.2.7}} = \boldsymbol{\varepsilon}_{\text{Lem.2.7}}(\ell), \quad \varepsilon_{\text{Lem.2.4}} = \boldsymbol{\varepsilon}_{\text{Lem.2.4}}(\ell), \quad \varepsilon_{\text{Fact 2.2}} = \boldsymbol{\varepsilon}_{\text{Fact 2.2}}(\ell). \tag{31}$$

We construct the corresponding (ℓ, t)-augmented (ω-weighted) 3-graph $\mathcal{A}$ ($\mathcal{A}^\omega$) for $\mathcal{H}'$ and $(\Pi, \mathscr{P})$ above using (16) ((18)) from Remark 2.15. Clearly, $\mathcal{A}$ and $\mathcal{A}^\omega$ are constructed in time $O(n^6)$. Indeed, for fixed $A \in \mathcal{M}$ (of which there are $|\mathcal{M}| = t^3\ell^3 \leq T_0^3 L_0^3 = O(1)$ many), testing $\alpha^A \geq \alpha_0$ takes time $O(n^3)$, and verifying Definition 2.3 (by greedy count) takes time $O(n^6)$. In Section 3.1, we shall prove that (cf. (4))

$$\frac{m^3}{\ell^3}\nu^*_{\mathcal{F}_0}(\mathcal{A}^\omega) \geq |\psi_{\Pi_0}| - (\alpha_0 + \delta + \tau + o(1))\, n^3 \overset{(30)}{\geq} (1-\xi)\nu^*_{\mathcal{F}_0}(\mathcal{H}) - (\alpha_0 + \delta + \tau + o(1))\, n^3, \tag{32}$$

where $o(1) \to 0$ as $n \to \infty$.

Step 2: Applying the Bounding Lemma. We now apply the Bounding Lemma (Lemma 2.18) to $\mathcal{A}^\omega$. To that end, with $\eta > 0$ in (19) and $\beta > 0$ in (21), we apply Lemma 2.18 to $\mathcal{A}^\omega$ to construct a β-bounded (cf. Definition 2.17) fractional $\mathcal{F}_0$-packing ψ_0 of $\mathcal{A}^\omega$ satisfying

$$|\psi_0| \geq \nu^*_{\mathcal{F}_0}(\mathcal{A}^\omega) - \eta\ell^3 t^3. \tag{33}$$

Recall from Lemma 2.18 that ψ_0 is constructed in time depending on $\ell \leq L_0 = O(1)$ and $t \leq T_0 = O(1)$.

Let us also define (construct) a few related objects. To begin, for $A \in \mathcal{A}$ (recall Definition 2.16), set

$$\binom{\mathcal{A}}{\mathcal{F}_0}^+ = \left\{ \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0} : \psi_0(\mathcal{F}) \geq \beta \right\} \quad \text{and} \quad \binom{\mathcal{A}}{\mathcal{F}_0}^+_A = \binom{\mathcal{A}}{\mathcal{F}_0}^+ \cap \binom{\mathcal{A}}{\mathcal{F}_0}_A. \tag{34}$$

For $A \in \mathcal{A}$ and $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+_A$, define

$$\sigma^A_{\mathcal{F}} = \psi_0(\mathcal{F}) \overset{(34)}{\geq} \beta. \tag{35}$$

Note that the sets and numbers above are constructed in time $O(1)$, since in an (ℓ, t)-augmented 3-graph $\mathcal{A}$, there are at most (recall $f = |V(\mathcal{F}_0)|$)

$$\left| \binom{\mathcal{A}}{\mathcal{F}_0} \right| \le t^f \ell^{3|\mathcal{F}_0|} \le T_0^f L_0^{3f^3} = O(1) \tag{36}$$

copies $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}$.

Step 3: Applying the Slicing Lemma. Fix $A \in \mathcal{A}$. With $\left\{ \sigma_{\mathcal{F}}^A : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A^+ \right\}$ from Step 2, we wish to apply the Slicing Lemma (Lemma 2.4) to $\mathcal{G} = \mathcal{H}^A$ (cf. (15)), but first check that it is appropriate to do so. Since $A \in \mathcal{A} = \omega^{-1}(0, 1]$ (cf. (18)), we have that $\mathcal{H}^A$ is (α^A, δ)-minimal w.r.t. P^A, where $\alpha^A \ge \alpha_0 = \beta$ (cf. (23). From (35), every $\sigma_{\mathcal{F}}^A$, $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A^+$, satisfies $\sigma_{\mathcal{F}}^A \ge \beta = \alpha_0$, and moreover,

$$\sum \left\{ \sigma_{\mathcal{F}}^A : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A^+ \right\} \overset{(35)}{=} \sum \left\{ \psi_0(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A^+ \right\} = \sum \left\{ \psi_0(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A \right\} \le \omega(A) \overset{(18)}{=} \alpha^A,$$

where the second equality holds on account that ψ_0 vanishes outside of $\binom{\mathcal{A}}{\mathcal{F}_0}_A^+$, and the inequality holds since ψ_0 is a fractional $\mathcal{F}_0$-packing of $\mathcal{A}^\omega$. From (24) and (25) (cf. (31)), we have that $\delta = \delta_{\text{Lem.2.4}}(\alpha_0, \delta' = \delta_{\text{Lem.2.7}})$ and $\varepsilon \le \varepsilon_{\text{Lem.2.4}}(\alpha_0, \delta' = \delta_{\text{Lem.2.7}}, \delta, 1/\ell)$ are appropriately chosen for an application of Lemma 2.4. Applying the Slicing Lemma to $\mathcal{G} = \mathcal{H}^A$, we construct, in time $O(m^3)$, a partition

$$\mathcal{H}^A = \mathcal{H}_0^A \cup \bigcup \left\{ \mathcal{H}_{\mathcal{F}}^A : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A^+ \right\} \tag{37}$$

satisfying that, for each $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A^+$, the 3-graph $\mathcal{H}_{\mathcal{F}}^A$ is $\left(\alpha_{\mathcal{F}}^A, \delta' = \delta_{\text{Lem.2.7}}\right)$-minimal with respect to P^A, where $\alpha_{\mathcal{F}}^A = \sigma_{\mathcal{F}}^A \pm \delta'$. Repeat over all at most $\binom{t}{3}\ell^3 \le T_0^3 L_0^3 = O(1)$ many $A \in \mathcal{A}$ in time $O(n^3)$.

Step 4: Applying the Packing Lemma. Fix $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+$. We apply the Packing Lemma to the following subhypergraph $\mathcal{G} = \mathcal{H}_{\mathcal{F}} \subset \mathcal{H}$:

$$V(\mathcal{H}_{\mathcal{F}}) = \bigcup_{i \in V(\mathcal{F})} V_i \qquad \text{and} \qquad \mathcal{H}_{\mathcal{F}} = \bigcup_{A \in \mathcal{F}} \mathcal{H}_{\mathcal{F}}^A. \tag{38}$$

Said differently, for each $1 \le i \le t$, we include $V_i \subset V(\mathcal{H}_{\mathcal{F}})$ if, and only if, $i \in V(\mathcal{F})$, and for each $A \in \mathcal{A}$, we take (recall (37))

$$\mathcal{H}_{\mathcal{F}} \cap \mathcal{K}_3\left(P^A\right) = \begin{cases} \mathcal{H}_{\mathcal{F}}^A & A \in \mathcal{F}, \\ \emptyset & A \in \mathcal{A} \setminus \mathcal{F}. \end{cases} \tag{39}$$

Let us check that it is appropriate to apply the Packing Lemma (Lemma 2.7) to $\mathcal{G} = \mathcal{H}_{\mathcal{F}}$.

We first confirm that $\mathcal{H}_{\mathcal{F}}$ meets the conditions of Setup 2.5 (the Packing Setup). For simplicity, but w.l.o.g., we assume $V(\mathcal{F}) = [f] \subset [t] = V(\mathcal{A})$. For the function $\phi_2 = \phi_2(\mathcal{F})$ in Definition 2.16, we write $a_{ij} = \phi_2(\{i, j\})$ for each $1 \le i < j \le f$. Then, the graph $P_{\mathcal{F}} = \bigcup \left\{ P_{a_{ij}}^{ij} : 1 \le i < j \le f \right\}$ underlies $\mathcal{H}_{\mathcal{F}}$, i.e., $\mathcal{H}_{\mathcal{F}} \subseteq \mathcal{K}_3(P_{\mathcal{F}})$, since for each $A = \{p_{a_{hi}}^{hi}, p_{b_{ij}}^{ij}, p_{c_{hj}}^{hj}\} \in \mathcal{F}$, we have by (15) and (37) (resp.) that $\mathcal{H}_{\mathcal{F}}^A \subseteq \mathcal{H}^A \subseteq \mathcal{K}_3(P^A)$. Note that $P_{\mathcal{F}}$ is an f-partite graph with vertex partition $V_1 \cup \cdots \cup V_f$, where $|V_1| = \cdots = |V_f| = m$ and where, for each $1 \le i < j \le f$, $P_{a_{ij}}^{ij}$ is (ℓ^{-1}, ε)-regular. Finally, suppose $A = \{p_{a_{hi}}^{hi}, p_{b_{ij}}^{ij}, p_{c_{hj}}^{hj}\} \in \mathcal{F}$. By (39), we see that $\mathcal{H}_{\mathcal{F}} \cap \mathcal{K}_3(P^A) = \mathcal{H}_{\mathcal{F}}^A$, which by Step 3 is $(\alpha_{\mathcal{F}}^A, \delta' = \delta_{\text{Lem.2.7}})$-minimal w.r.t. P^A. Moreover, by the Slicing Lemma, $\alpha_{\mathcal{F}}^A = \sigma_{\mathcal{F}}^A \pm \delta'$, where $\sigma_{\mathcal{F}}^A \overset{(35)}{=} \psi_0(\mathcal{F}) \ge \beta \overset{(23)}{=} \alpha_0$ is constant and bounded over all $A \in \mathcal{F}$. (In other words, $\psi_0(\mathcal{F})$ plays the role of α in Setup 2.5.) Note that our constants are also chosen appropriately for an application of the Packing Lemma (Lemma 2.7). Indeed, for α_0 above and $\rho > 0$ in (19), we defined $\delta' = \delta_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho) > 0$ (cf. (24)) to be the constant guaranteed by the Packing Lemma, and we defined $\varepsilon \le \varepsilon_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho, \delta', 1/\ell)$ in (25) (cf. (31)) to be appropriate for an application of the Packing Lemma.

Applying Lemma 2.7 to $\mathcal{G} = \mathcal{H}_{\mathcal{F}}$ and $P = P_{\mathcal{F}}$ above, we construct, in time polynomial in m, an $\mathcal{F}_0$-packing $\mathscr{F}_{\mathcal{H}_{\mathcal{F}}}$ of $\mathcal{H}_{\mathcal{F}}$ of size

$$|\mathscr{F}_{\mathcal{H}_{\mathcal{F}}}| \geq (1-2\rho)\psi_0(\mathcal{F})\frac{m^3}{\ell^3}, \tag{40}$$

where every element of $\mathscr{F}_{\mathcal{H}_{\mathcal{F}}}$ is a partite-isomorphic copy of $\mathcal{F}$ (cf. Definition 2.6). Repeat over all $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+$, of which there are at most $O(1)$ many (cf. (36)).

Output. Construct the family $\mathscr{F}_{\mathcal{H}} = \bigcup \left\{ \mathscr{F}_{\mathcal{H}_{\mathcal{F}}} : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+ \right\}$ in time $O(n^3)$. (That is, collect $O(m^3)$ copies of $\mathcal{F}_0$ over at most $O(1)$ indices $\mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+$ (cf. (36)).

It remains to check the correctness of the algorithm. To that end, the family $\mathscr{F}_{\mathcal{H}}$ was clearly constructed in time polynomial in n. Regarding the remaining details, we first prove that $\mathscr{F}_{\mathcal{H}}$ is an $\mathcal{F}_0$-packing of $\mathcal{H}$.

Proof that $\mathscr{F}_{\mathcal{H}}$ is an $\mathcal{F}_0$-packing of $\mathcal{H}$. Let $\mathcal{F}, \mathcal{F}' \in \mathscr{F}_{\mathcal{H}}$, and for contradiction, suppose $\mathcal{F} \cap \mathcal{F}' \neq \emptyset$. By construction, there exist $\hat{\mathcal{F}}, \hat{\mathcal{F}}' \in \binom{\mathcal{A}}{\mathcal{F}_0}^+$ so that $\mathcal{F} \in \mathscr{F}_{\mathcal{H}_{\hat{\mathcal{F}}}}$ and $\mathcal{F}' \in \mathscr{F}_{\mathcal{H}_{\hat{\mathcal{F}}'}}$. If $\hat{\mathcal{F}} = \hat{\mathcal{F}}'$, then $\mathscr{F}_{\mathcal{H}_{\hat{\mathcal{F}}}} = \mathscr{F}_{\mathcal{H}_{\hat{\mathcal{F}}'}}$, and so $\mathcal{F} \cap \mathcal{F}' \neq \emptyset$ contradicts the Packing Lemma (which ensured that $\mathscr{F}_{\mathcal{H}_{\hat{\mathcal{F}}}} = \mathscr{F}_{\mathcal{H}_{\hat{\mathcal{F}}'}}$ was an $\mathcal{F}_0$-packing of $\mathcal{H}_{\hat{\mathcal{F}}} = \mathcal{H}_{\hat{\mathcal{F}}'}$). Henceforth, we assume $\hat{\mathcal{F}} \neq \hat{\mathcal{F}}'$.

Fix $\{x, y, z\} \in \mathcal{F} \cap \mathcal{F}'$. Clearly, this implies (cf. (38), (39)),

$$\{x, y, z\} \in \mathcal{H}_{\hat{\mathcal{F}}} \quad \text{and} \quad \{x, y, z\} \in \mathcal{H}_{\hat{\mathcal{F}}'}. \tag{41}$$

Since $\mathcal{F}, \mathcal{F}' \subset \mathcal{H}'$ (recall the notation in Theorem 2.12), there exist $1 \leq h < i < j \leq t$ and $1 \leq a, b, c \leq \ell$ so that

$$\{x, y, z\} \in \mathcal{K}_3(P_a^{hi} \cup P_b^{ij} \cup P_c^{hj}). \tag{42}$$

Write $A = \{p_a^{hi}, p_b^{ij}, p_c^{hj}\} \in \mathcal{A}$. By (39), we have

$$\mathcal{H}_{\hat{\mathcal{F}}} \cap \mathcal{K}_3(P^A) = \mathcal{H}_{\hat{\mathcal{F}}}^A \quad \text{and} \quad \mathcal{H}_{\hat{\mathcal{F}}'} \cap \mathcal{K}_3(P^A) = \mathcal{H}_{\hat{\mathcal{F}}'}^A. \tag{43}$$

But now, (41)–(43) imply that $\{x, y, z\} \in \mathcal{H}_{\hat{\mathcal{F}}}^A \cap \mathcal{H}_{\hat{\mathcal{F}}'}^A$, which contradicts the Slicing Lemma. (Indeed, $\mathcal{H}_{\hat{\mathcal{F}}}^A$ and $\mathcal{H}_{\hat{\mathcal{F}}'}^A$ are distinct classes (since $\hat{\mathcal{F}} \neq \hat{\mathcal{F}}'$) of a partition.) □

Proof that $\mathscr{F}_{\mathcal{H}}$ has size promised in (29). By construction, we have

$$|\mathscr{F}_{\mathcal{H}}| = \sum \left\{ |\mathscr{F}_{\mathcal{H}_{\mathcal{F}}}| : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+ \right\} \overset{(40)}{\geq} (1-2\rho)\frac{m^3}{\ell^3} \sum \left\{ \psi_0(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{A}}{\mathcal{F}_0}^+ \right\} \geq (1-2\rho)\frac{m^3}{\ell^3}\,|\psi_0|$$

$$\overset{(33)}{\geq} (1-2\rho)\frac{m^3}{\ell^3}\left(\nu_{\mathcal{F}_0}^*(\mathcal{A}^\omega) - \eta t^3 \ell^3\right) \overset{(32)}{\geq} (1-2\rho)\left((1-\xi)\nu_{\mathcal{F}_0}^*(\mathcal{H}) - (\alpha_0 + \delta + \tau + o(1))n^3 - \eta n^3\right)$$

$$\geq \nu_{\mathcal{F}_0}^*(\mathcal{H}) - (\xi + \alpha_0 + \delta + \tau + \eta + 2\rho + o(1))\,n^3 \overset{(22),(26)}{\geq} \nu_{\mathcal{F}_0}^*(\mathcal{H}) - 8\xi n^3 \overset{(19)}{=} \nu_{\mathcal{F}_0}^*(\mathcal{H}) - \zeta n^3,$$

as promised. □

All that remains to prove Theorem 1.3 is the proof of (32).

3.1. **Proof of (32).** It suffices to produce a fractional $\mathcal{F}_0$-packing $\tilde{\psi}_0 : \binom{\mathcal{A}^\omega}{\mathcal{F}_0} \to [0,1]$ for which $m^3|\tilde{\psi}_0|/\ell^3$ has the lower bound promised in (32). To that end, we establish some notation and terminology. Write $\binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}$ to denote the copies $\mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0}$ (cf. Lemma 1.7) for which $\mathcal{F} \subset \mathcal{H}'$ (cf. Theorem 2.12), and fix $\mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}$. We define the following $\tilde{\mathcal{F}} = (\tilde{\phi}_1, \tilde{\phi}_2) \in \binom{\mathcal{A}}{\mathcal{F}_0}$ to be the *projection* of $\mathcal{F}$ onto $\mathcal{A}$. Define $\tilde{\phi}_1 : V(\mathcal{F}) \to [t]$ by $\tilde{\phi}_1(v) = i$, if, and only if, $v \in V_i$. Then $\tilde{\phi}_1$ is an injection since $\mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}$ crosses Π_0 (cf. Lemma 1.7 and Remark 2.13). Write $\tilde{\Phi}_1 = \tilde{\phi}_1(V(\mathcal{F}))$, and for $i \in \tilde{\Phi}_1$, write $v_i = \tilde{\phi}_1^{-1}(i)$. Define

$\tilde{\phi}_2 : \binom{\tilde{\Phi}_1}{2} \to [\ell]$ by $\tilde{\phi}_2(\{i,j\}) = a$ if, and only if, $\{v_i, v_j\} \in P_a^{ij}$. ($\tilde{\phi}_2$ is well-defined since $\{v_i, v_j\} \in K[V_i, V_j]$ (cf. Definition 2.10).) To check that $\tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}$ (cf. Definition 2.16), fix $\{u,v,w\} \in \mathcal{F}$ and write

$$\left(\tilde{\phi}_1(u), \tilde{\phi}_1(v), \tilde{\phi}_1(w)\right) = (h,i,j) \quad \text{and} \quad \left(\tilde{\phi}_2(\{h,i\}), \tilde{\phi}_2(\{i,j\}), \tilde{\phi}_2(\{h,j\})\right) = (a,b,c).$$

We show that $A = \{p_a^{hi}, p_b^{ij}, p_c^{hj}\} \in \mathcal{A}$. To that end, by construction we have that $\{u,v,w\} \in \mathcal{K}_3(P^A)$. Since $\{u,v,w\} \in \mathcal{F} \subset \mathcal{H}'$, the Regularity Lemma guarantees $\mathcal{H}^A$ (cf. (15)) is (α^A, δ)-minimal w.r.t. $\mathcal{K}_3(P^A)$ for some $\alpha^A \geq \alpha_0$. Then by (18), $A \in \mathcal{A}$.

Set (cf. (14))

$$\Delta = \max\left\{|\mathcal{K}_3(P^A)| : A \in \mathcal{A}\right\}. \tag{44}$$

Define the function $\tilde{\psi}_0 : \binom{\mathcal{A}}{\mathcal{F}_0} \to [0,1]$ by the rule

$$\tilde{\psi}_0(\tilde{\mathcal{F}}) = \frac{1}{\Delta} \sum \left\{ \psi(F) : F \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0} \text{ has projection } \tilde{\mathcal{F}} \right\}. \tag{45}$$

To show that $\tilde{\psi}_0$ is a fractional $\mathcal{F}_0$-packing of $\mathcal{A}^\omega$, fix $A = \{p_a^{hi}, p_b^{ij}, p_c^{hj}\} \in \mathcal{A}$. Then (cf. Definition 2.16)

$$\begin{aligned}
&\sum \left\{ \tilde{\psi}_0(\tilde{\mathcal{F}}) : \tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A \right\} \\
&\qquad = \frac{1}{\Delta} \sum \left\{ \sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0} \text{ has projection } \tilde{\mathcal{F}} \right\} : \tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A \right\} \\
&\qquad \overset{(44)}{\leq} \frac{1}{|\mathcal{K}_3(P^A)|} \sum \left\{ \sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0} \text{ has projection } \tilde{\mathcal{F}} \right\} : \tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A \right\}.
\end{aligned}$$

Note that an element $\mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}$ projects to some element $\tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A$ if, and only if, $\mathcal{F} \cap \mathcal{H}^A \neq \emptyset$ (cf. (15)). Therefore,

$$\begin{aligned}
\sum \left\{ \tilde{\psi}_0(\tilde{\mathcal{F}}) : \tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}_A \right\} &\leq \frac{1}{|\mathcal{K}_3(P^A)|} \sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0} \text{ satisfies } \mathcal{F} \cap \mathcal{H}^A \neq \emptyset \right\} \\
&= \frac{1}{|\mathcal{K}_3(P^A)|} \sum \left\{ \sum \left\{ \psi(\mathcal{F}) : \{u,v,w\} \in \mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0} \right\} : \{u,v,w\} \in \mathcal{H}^A \right\} \\
&\leq \frac{1}{|\mathcal{K}_3(P^A)|} \sum \left\{ \sum \left\{ \psi(\mathcal{F}) : \{u,v,w\} \in \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0} \right\} : \{u,v,w\} \in \mathcal{H}^A \right\} \\
&\overset{\text{cf.}(2)}{=} \frac{1}{|\mathcal{K}_3(P^A)|} \sum \left\{ \sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\{u,v,w\}} \right\} : \{u,v,w\} \in \mathcal{H}^A \right\} \\
&\leq \frac{|\mathcal{H}^A|}{|\mathcal{K}_3(P^A)|} = d_{\mathcal{H}}(P) = \alpha^A = \omega(A),
\end{aligned}$$

where in the last inequality, we used that ψ is a fractional $\mathcal{F}_0$-packing of $\mathcal{H}$, and in the last equality, we used that $A \in \mathcal{A}$ and (16) and (18).

To conclude the proof of (32), consider the quality $|\psi_{\Pi_0}| - \Delta|\tilde{\psi}_0|$. From (45), we see that $\Delta|\tilde{\psi}_0|$ equals

$$\sum \left\{ \sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0} \text{ has projection } \tilde{\mathcal{F}} \right\} : \tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0} \right\} = \sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0} \right\},$$

since every $\mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}$ projects to a unique $\tilde{\mathcal{F}} \in \binom{\mathcal{A}}{\mathcal{F}_0}$. Thus,

$$|\psi_{\Pi_0}| - \Delta \left|\tilde{\psi}_0\right| = \sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0} \right\} - \sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0} \right\}$$
$$= \sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0} \setminus \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0} \right\}.$$

Now, $\mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0} \setminus \binom{\mathcal{H}'}{\mathcal{F}_0}_{\Pi_0}$ if, and only if, there exists $\{x, y, z\} \in \mathcal{F} \cap (\mathcal{H} \setminus \mathcal{H}')$ which crosses Π_0. Thus,

$$|\psi_{\Pi_0}| - \Delta \left|\tilde{\psi}_0\right| \leq \sum \left\{ \sum \left\{ \psi(\mathcal{F}) : \{x, y, z\} \in \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\Pi_0} \right\} : \{x, y, z\} \in \mathcal{H} \setminus \mathcal{H}' \text{ crosses } \Pi_0 \right\}$$
$$\leq \sum \left\{ \sum \left\{ \psi(\mathcal{F}) : \{x, y, z\} \in \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0} \right\} : \{x, y, z\} \in \mathcal{H} \setminus \mathcal{H}' \text{ crosses } \Pi_0 \right\}$$
$$= \sum \left\{ \sum \left\{ \psi(\mathcal{F}) : \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}_{\{x,y,z\}} \right\} : \{x, y, z\} \in \mathcal{H} \setminus \mathcal{H}' \text{ crosses } \Pi_0 \right\}$$
$$\leq |\{\{x, y, z\} : \{x, y, z\} \in \mathcal{H} \setminus \mathcal{H}' \text{ crosses } \Pi_0\}| \leq |\mathcal{H} \setminus \mathcal{H}'| \overset{\text{Thm.2.12}}{\leq} (\alpha_0 + \delta + o(1))\, n^3,$$

where in the third to last inequality, we used that ψ is a fractional $\mathcal{F}_0$-packing of $\mathcal{H}$. Using Fact 2.2, the Triangle Counting Lemma (cf. (19), (25), (31)), the bounds above imply

$$(1 + \tau)\frac{m^3}{\ell^3}\nu^*_{\mathcal{F}_0}(\mathcal{A}^\omega) \geq \Delta\nu^*_{\mathcal{F}_0}(\mathcal{A}^\omega) \geq \Delta \left|\tilde{\psi}_0\right| \geq |\psi_{\Pi_0}| - (\alpha_0 + \delta + o(1))\, n^3,$$

and so

$$\frac{m^3}{\ell^3}\nu^*_{\mathcal{F}_0}(\mathcal{A}^\omega) \geq \frac{1}{1+\tau}\left(|\psi_{\Pi_0}| - (\alpha_0 + \delta + o(1))\, n^3\right) \geq |\psi_{\Pi_0}| - (\alpha_0 + \delta + \tau + o(1))\, n^3,$$

as promised.

4. Proof of the Packing Lemma (Lemma 2.7)

The proof of the Packing Lemma (Lemma 2.7) is based on the following well-known result of Grable [7] on matchings. For a j-uniform hypergraph $\mathcal{J}$, a *matching* $\mathscr{J}$ in $\mathcal{J}$ is a family of pairwise disjoint edges from $\mathcal{J}$. As usual, for $x, x' \in V(\mathcal{J})$, set $N_{\mathcal{J}}(x) = \{I : I \cup \{x\} \in \mathcal{J}\}$, $N_{\mathcal{J}}(x, x') = N_{\mathcal{J}}(x) \cap N_{\mathcal{J}}(x')$, $\deg_{\mathcal{J}}(x) = |N_{\mathcal{J}}(x)|$, and $\deg_{\mathcal{J}}(x, x') = |N_{\mathcal{J}}(x, x')|$.

Theorem 4.1 (Grable [7]). *Let $j \geq 2$ be a fixed integer, and let $C > 1$ be fixed. For all $\lambda > 0$, there exists $\beta = \beta_{\text{Thm.4.1}}(j, C, \lambda) > 0$ so that the following holds. For a sufficiently large $\Delta > 0$ and a sufficiently large vertex set X, let $\mathcal{J}$ be a j-graph on the vertex set X which satisfies the following conditions:*

(0) *all distinct pairs of vertices $x \neq x' \in X$ satisfy* $\deg_{\mathcal{J}}(x, x') < \beta\Delta$*;*
(1) *all vertices $x \in X$ satisfy* $\deg_{\mathcal{J}}(x) \leq C\Delta$*;*
(2) *all but $\beta|X|$ vertices $x \in X$ satisfy* $\deg_{\mathcal{J}}(x) = (1 \pm \beta)\Delta$*.*

Then, there exists a matching $\mathscr{J}$ of $\mathcal{J}$ covering all but $\lambda|X|$ vertices of X. Moreover, $\mathscr{J}$ can be constructed in time polynomial in $|X|$.

When we apply Theorem 4.1, no effort will be needed to verify Condition (0) of its hypothesis. To verify Conditions (1) and (2), we will use the following *Extension Lemmas*, which have appeared in various forms in the regularity literature. In particular, Condition (1) will be handled with the following graph extension lemma, stated here for triangles, whose standard proof we omit for simplicity.

Lemma 4.2 (triangle extension lemma)**.** *For all integers $f \geq 2$, and for all $d, \mu > 0$, there exists $\varepsilon = \varepsilon_{\text{Lem.4.2}}(f, d, \mu) > 0$ so that the following holds. Let P be a graph satisfying the hypothesis of Setup 2.5 with $d, \varepsilon > 0$ above, and with a sufficiently large integer m. For each $1 \leq a < b < c \leq f$, all but $\mu|\mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac})|$ triangles $\{u, v, w\} \in \mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac})$ belong to within $(1 \pm (1/2)) d^{\binom{f}{2} - 3} m^{f-3}$ many cliques K_f in P.*

Handling Condition (2) in the hypothesis of Theorem 4.1 is the delicate part of our application. Our work will largely be based on the following hypergraph analogue of Lemma 4.2, stated here for triples.

Lemma 4.3 (triple extension lemma)**.** *Let $\mathcal{F}_0$ be a fixed 3-graph with $V(\mathcal{F}_0) = [f]$. For all $\alpha_0, \zeta > 0$, there exists $\delta = \delta_{\text{Lem.4.3}}(\mathcal{F}_0, \alpha_0, \zeta) > 0$ so that, for all $d^{-1} \in \mathbb{N}$, there exists $\varepsilon = \varepsilon_{\text{Lem.4.3}}(\mathcal{F}_0, \alpha_0, \zeta, \delta, d) > 0$ so that the following holds.*

Let P and $\mathcal{G}$ satisfy the hypothesis of Setup 2.5 with $\mathcal{F}_0$ and some $\alpha \geq \alpha_0$, with $\delta, d, \varepsilon > 0$ above, and with a sufficiently large integer m. Then, all but $\zeta|\mathcal{G}|$ triples $\{u, v, w\} \in \mathcal{G}$ belong to within $(1 \pm \zeta)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3}$ many partite-isomorphic copies of $\mathcal{F}_0$ in $\mathcal{G}$.

Lemma 4.3 is a non-trivial result which was essentially proven by Haxell et al. [10], but for a concept of hypergraph regularity different from that of Setup 2.5. Lemma 4.3 was later generalized to a k-uniform setting by Cooley et al. [2], but again for a different concept of hypergraph regularity. For completeness, we give a proof of Lemma 4.3 in the next section.

4.1. Proof of Lemma 2.7.

We begin our proof of Lemma 2.7 by discussing the constants promised by it. Afterward, we pause to prepare our application of Theorem 4.1 to the setting of Lemma 2.7.

4.1.1. *Constants for Lemma 2.7.* Let $\mathcal{F}_0$ be a fixed 3-graph with $V(\mathcal{F}_0) = [f]$, and let $\alpha_0, \rho > 0$ be given. To define the constant $\delta = \delta_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho) > 0$ promised by Lemma 2.7, we consider some auxiliary constants. In the context of Theorem 4.1, set

$$j = |\mathcal{F}_0|, \quad C = 2\alpha_0^{1-|\mathcal{F}_0|}, \quad \text{and} \quad \lambda = \rho/2. \tag{46}$$

Theorem 4.1 guarantees the constant

$$\beta = \beta_{\text{Thm.4.1}}(j, C, \lambda) > 0, \quad \text{where for convenience we also define } \zeta = \beta/4. \tag{47}$$

With $\mathcal{F}_0$ and α_0 given above, and with ζ defined in (47), Lemma 4.3 guarantees the constant $\delta_{\text{Lem.4.3}} = \delta_{\text{Lem.4.3}}(\mathcal{F}_0, \alpha_0, \zeta) > 0$. We define the promised constant $\delta = \delta_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho) > 0$ by

$$\delta = \min\{\alpha_0\rho/6,\ \delta_{\text{Lem.4.3}}\}. \tag{48}$$

Let $d^{-1} \in \mathbb{N}$ be given. To define the constant $\varepsilon = \varepsilon_{\text{Lem.2.7}}(\mathcal{F}_0, \alpha_0, \rho, \delta, d) > 0$ promised by Lemma 2.7, we again define some auxiliary constants. First, with $d > 0$ given above and with $\tau = 1/2$, let $\varepsilon_{\text{Fact 2.2}}(d, 1/2) > 0$ be the constant guaranteed by the triangle counting lemma (Fact 2.2). Second, with the parameters $\mathcal{F}_0, \alpha_0, d$ given above, and with $\zeta, \delta > 0$ defined in (47) and (48), let

$$\varepsilon_{\text{Lem.4.3}} = \varepsilon_{\text{Lem.4.3}}(\mathcal{F}_0, \alpha_0, \zeta, \delta, d) > 0$$

be the constant guaranteed by Lemma 4.3. Finally, with all parameters from (46)–(48), set

$$\mu = \min\left\{\delta,\ \tfrac{1}{2j}\zeta\alpha_0^{|\mathcal{F}_0|} d^{\binom{f}{2}} (\alpha_0 - \delta)(1 - (\rho/2))\beta\right\}, \tag{49}$$

and let $\varepsilon_{\text{Lem.4.2}} = \varepsilon_{\text{Lem.4.2}}(f, d, \mu) > 0$ be the constant guaranteed by Lemma 4.2. We define

$$\varepsilon = \min\left\{\varepsilon_{\text{Fact 2.2}},\ \varepsilon_{\text{Lem.4.2}},\ \varepsilon_{\text{Lem.4.3}}\right\}. \tag{50}$$

In all that follows, we take the integer m to be sufficiently large, whenever needed.

Now, let P and $\mathcal{G}$ satisfy the hypothesis of Setup 2.5 with $\mathcal{F}_0$ fixed above, with some $\alpha \geq \alpha_0$, with $\delta, d, \varepsilon > 0$ fixed above, and with a sufficiently large integer m. To construct the desired $\mathcal{F}_0$-packing $\mathscr{F} = \mathscr{F}_{\mathcal{G}}$ of $\mathcal{G}$, we will apply Theorem 4.1, for which we need to prepare. Once prepared, the output of Theorem 4.1 will rather easily deliver the promised family $\mathscr{F}$ (see upcoming Remarks 4.4 and 4.6).

4.1.2. *Preparations for Theorem 4.1.* Recall from (46) and (47) that we already prepared constants $j, C, \lambda, \beta > 0$ for an application of Theorem 4.1. The goal of the current preparations is to define a parameter Δ, a vertex set X, and a j-graph $\mathcal{J}$, to which we can apply Theorem 4.1 (with C, λ, β above). Defining Δ is straightforward. For that, recall that $\alpha \geq \alpha_0 > 0$ and $d > 0$ were given above, as was the sufficiently large integer $m \in \mathbb{N}$. We correspondingly define

$$\Delta = \alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} = \Omega(m^{f-3}), \tag{51}$$

which can be taken sufficiently large whenever needed (as required in the hypothesis of Theorem 4.1). Defining X and $\mathcal{J}$ is less straightforward. For that, we first define a superset of vertices $W \supseteq X$ and a (super)hypergraph $\tilde{\mathcal{J}} \supseteq \mathcal{J}$, where $\tilde{\mathcal{J}}$ will have vertex set W. Once W and $\tilde{\mathcal{J}}$ are defined, we will extract $X \subseteq W$, and we will define $\mathcal{J} = \tilde{\mathcal{J}}[X]$ to be the subhypergraph of $\tilde{\mathcal{J}}$ induced on X.

Define $W = \mathcal{G}$ to be the triples of $\mathcal{G}$. We take W to be the vertex set of the following j-uniform hypergraph $\tilde{\mathcal{J}}$, where $j = |\mathcal{F}_0|$ (cf. (46)) is the number of triples of $\mathcal{F}_0$: for each $\{e_1, \ldots, e_j\} \in \binom{W}{j}$, put

$$\{e_1, \ldots, e_j\} \in \tilde{\mathcal{J}} \quad \Longleftrightarrow \quad \{e_1, \ldots, e_j\} \text{ is the edge-set of a partite-isomorphic copy of } \mathcal{F}_0 \text{ in } \mathcal{G}. \tag{52}$$

Since $\mathcal{G}$ is f-partite, $\tilde{\mathcal{J}}$ may be constructed on W in time $O(m^f)$. We now pause our preparations to comment on some of the formal strategy of our proof.

Remark 4.4. Observe that a matching $\mathscr{J}$ of $\tilde{\mathcal{J}}$ corresponds to an $\mathcal{F}_0$-packing $\mathscr{F} = \mathscr{F}(\mathscr{J})$ of $\mathcal{G}$, where $\mathscr{F}$ consists of partite-isomorphic copies of $\mathcal{F}_0$ in $\mathcal{G}$. Indeed, by construction in (52), each element $J \in \mathscr{J}$ defines the edge-set of a partite-isomorphic copy of $\mathcal{F}_0$ in $\mathcal{G}$. Moreover, since the elements of $\mathscr{J}$ are pairwise disjoint, the elements of $\mathscr{F}$ are pairwise edge-disjoint.

Clearly, the correspondence $\mathscr{J} \mapsto \mathscr{F}$ preserves cardinality, i.e., $|\mathscr{J}| = |\mathscr{F}|$. As such, our plan is to use Theorem 4.1 to efficiently construct a 'large' matching $\mathscr{J}_0$ of $\tilde{\mathcal{J}}$, which correspondingly builds a 'large' $\mathcal{F}_0$-packing $\mathscr{F}_0 = \mathscr{F}(\mathscr{J}_0)$ of $\mathcal{G}$. Unfortunately, Theorem 4.1 will not quite (at least directly) apply to $\tilde{\mathcal{J}}$. Fortunately, we can delete a small number of vertices $w \in W$ to obtain a set $X \subseteq W$ and an induced subhypergraph $\mathcal{J} = \tilde{\mathcal{J}}[X]$ to which Theorem 4.1 will apply. We continue with these preparations. □

Define the promised vertex set X by

$$X = \left\{ e \in W : \deg_{\tilde{\mathcal{J}}}(e) \leq 2d^{\binom{f}{2}-3} m^{f-3} \right\}. \tag{53}$$

We view the vertices of X as the set of elements $e \in W$ whose $\tilde{\mathcal{J}}$-degree isn't 'too large'. Note that, since $\mathcal{G}$ is f-partite, we can identify $X \subseteq W$ in time $O(m^f)$. We already hinted in Remark 4.4 that we expect $|W \setminus X|$ to be small, which we now formally state and prove.

Claim 4.5. $|X| \geq (1 - (\rho/2))|W|$.

Proof of Claim 4.5. Indeed, Setup 2.5 gives

$$|W| = |\mathcal{G}| = \sum_{\{a,b,c\} \in \mathcal{F}_0} \alpha_{abc} |\mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac})| = (\alpha \pm \delta) \sum_{\{a,b,c\} \in \mathcal{F}_0} |\mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac})|. \tag{54}$$

By definition, each element $e \in W \setminus X$ (as an edge of $\mathcal{G}$) belongs to 'too many' partite-isomorphic copies of $\mathcal{F}_0$ in $\mathcal{G}$. Consequently, each element $e \in W \setminus X$ (as a triangle in P) belongs to 'too many' copies of K_f in P, and the number of such triangles can be bounded using Lemma 4.2. To that end, for each $\{a, b, c\} \in \mathcal{F}_0$, we apply Lemma 4.2 (cf. (50)) to $\mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac})$ to conclude

$$\begin{aligned} |X| &\geq |W| - \mu \sum_{\{a,b,c\} \in \mathcal{F}_0} |\mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac})| \overset{(54)}{\geq} (\alpha - \delta - \mu) \sum_{\{a,b,c\} \in \mathcal{F}_0} |\mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac})| \\ &\overset{(49),(54)}{\geq} \frac{\alpha - 2\delta}{\alpha + \delta} |W| \geq \frac{1 - 2(\delta/\alpha_0)}{1 + (\delta/\alpha_0)} |W| \geq (1 - 3(\delta/\alpha_0))|W| \overset{(48)}{\geq} (1 - (\rho/2))|W|, \end{aligned} \tag{55}$$

as promised. □

Define the promised subhypergraph $\mathcal{J}$ by $\mathcal{J} = \tilde{\mathcal{J}}[X]$, which is the subhypergraph of $\tilde{\mathcal{J}}$ induced on the set X. Equivalently, $\mathcal{J}$ is the hypergraph obtained by deleting the vertices of $W \setminus X$ from $\tilde{\mathcal{J}}$. We now continue developing our line of strategy from Remark 4.4.

Remark 4.6. We will show that Theorem 4.1 applies to the j-graph $\mathcal{J}$ on vertex set X (with the parameter Δ defined in (51), and with the constants $C, \lambda, \beta > 0$ in (46) and (47)). Applying Theorem 4.1 to $\mathcal{J}$ will be easily conducted below (in the context of the current Remark 4.6). However, the *applicability* of Theorem 4.1 to $\mathcal{J}$ is not entirely immediate, and the remainder of the section is devoted to its proof. In particular, while deleting $W \setminus X$ from $\tilde{\mathcal{J}}$ will mostly have clear benefits, some care will be needed in places (concerning Condition (2) of Theorem 4.1) to ensure that not too much structure of $\tilde{\mathcal{J}}$ is lost to these deleted vertices.

The successful application of Theorem 4.1 to $\mathcal{J}$ polynomially constructs a matching $\mathscr{J}_0$ in $\mathcal{J}$

$$\textit{which covers all but } \lambda|X| \overset{(46)}{=} (\rho/2)|X| \textit{ vertices } x \in X. \tag{56}$$

As such, Claim 4.5 and (56) combine to say that $\mathscr{J}_0$ in $\mathcal{J}$ covers all but $(\rho/2)|X| + (\rho/2)|W| \leq \rho|W|$ vertices $w \in W$. Remark 4.4 says that the corresponding $\mathcal{F}_0$-packing $\mathscr{F}_0 = \mathscr{F}(\mathscr{J}_0)$ covers all but $\rho|\mathcal{G}|$ edges of $\mathcal{G}$, as promised by Lemma 2.7. Thus, it remains only to show that Theorem 4.1 applies to $\mathcal{J}$. □

4.1.3. *Applicability of Theorem 4.1 to $\mathcal{J}$.* We verify that Theorem 4.1 applies to Δ, X, and $\mathcal{J}$ from (51) and (53) with the constants $C, \lambda, \beta > 0$ from (46) and (47). To that end, we already noted (cf. (51)) that Δ can be taken sufficiently large whenever needed. Similarly, Claim 4.5 shows that $|X| \geq (1 - (\rho/2))|W| = \Omega(|\mathcal{G}|) = \Omega(m^3)$ can be taken sufficiently large whenever needed. Condition (0) in the hypothesis of Theorem 4.1 is immediately satisfied by Δ, X, and $\mathcal{J}$. Indeed, as distinct triples, every pair $e \neq e' \in X$ satisfies $|e \cup e'| \geq 4$, and so

$$\deg_{\mathcal{J}}(e, e') \leq \deg_{\tilde{\mathcal{J}}}(e, e') \leq m^{f-4} = o(m^{f-3}) \overset{(47),\,(51)}{\leq} \beta\Delta.$$

Moreover, Condition (1) in the hypothesis of Theorem 4.1 is also satisfied by Δ, X, and $\mathcal{J}$. Indeed, it follows by the definition of X that every element $e \in X$ satisfies

$$\deg_{\mathcal{J}}(e) \leq \deg_{\tilde{\mathcal{J}}}(e) \leq 2d^{\binom{f}{2}-3} m^{f-3} \overset{(51)}{=} 2\alpha^{1-|\mathcal{F}_0|}\Delta \leq 2\alpha_0^{1-|\mathcal{F}_0|}\Delta \overset{(46)}{=} C\Delta,$$

where we used $\alpha \geq \alpha_0$. It remains only to verify that Δ, X, and $\mathcal{J}$ satisfy Condition (2) in the hypothesis of Theorem 4.1.

With $\zeta = \beta/4$ from (47), define

$$Y = \left\{ e \in X : \deg_{\tilde{\mathcal{J}}}(e) = (1 \pm \zeta)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} \right\}. \tag{57}$$

Note that Y is defined in terms of degrees in $\tilde{\mathcal{J}}$, before $W \setminus X$ is deleted. Lemma 4.3 (and our choice of constants in (47)–(50)) guarantees that Y is a 'large' subset of X:

$$|Y| \geq |X| - \zeta|W| \overset{(55)}{\geq} \left(1 - \frac{\zeta}{1 - (\rho/2)}\right)|X| \geq (1 - 2\zeta)\,|X| \overset{(47)}{=} (1 - (\beta/2))\,|X|. \tag{58}$$

Now, by the definition of Y in (57), every element $e \in Y$ satisfies

$$\deg_{\mathcal{J}}(e) \leq \deg_{\tilde{\mathcal{J}}}(e) \leq (1 + \zeta)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} \overset{(47),\,(51)}{<} (1 + \beta)\Delta. \tag{59}$$

However, it is not clear to what extent (if any) a lower bound on $\deg_{\mathcal{J}}(e)$ matches. In particular, and in principle, deleting $W \setminus X$ might make $\deg_{\mathcal{J}}(e) = 0$, even though $\deg_{\tilde{\mathcal{J}}}(e)$ bad been large. When so, we will show that this won't happen often.

Continuing, and to complement (59), consider the following hypergraphs:

$$\overline{\mathcal{J}} = \tilde{\mathcal{J}} \setminus \mathcal{J} = \left\{ J \in \tilde{\mathcal{J}} : J \cap (W \setminus X) \neq \emptyset \right\}, \quad \text{and for each } e \in W, \text{ set} \quad \overline{\mathcal{J}}_e = \left\{ J \in \overline{\mathcal{J}} : e \in J \right\}.$$

Then, $\overline{\mathcal{J}}$ is the complement of $\mathcal{J}$ inside of $\tilde{\mathcal{J}}$, and for each $e \in W$, the hypergraph $\overline{\mathcal{J}}_e$ consists of all $J \in \overline{\mathcal{J}}$ containing e. Now, for every $e \in Y$, we have that

$$\deg_{\mathcal{J}}(e) = \deg_{\tilde{\mathcal{J}}}(e) - |\overline{\mathcal{J}}_e| \overset{(57)}{\geq} (1-\zeta)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} - |\overline{\mathcal{J}}_e|.$$

Thus, define

$$Y_{\text{good}} = \left\{ e \in Y : |\overline{\mathcal{J}}_e| \leq \zeta \alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} \right\} \quad \text{and} \quad Y_{\text{bad}} = Y \setminus Y_{\text{good}}. \tag{60}$$

Then, every element $e \in Y_{\text{good}}$ satisfies

$$\deg_{\mathcal{J}}(e) \geq (1-2\zeta)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} \overset{(51)}{=} (1-2\zeta)\Delta,$$

and so with $\beta = 4\zeta$ from (47), and with (59), every element $e \in Y_{\text{good}}$ satisfies $\deg_{\mathcal{J}}(e) = (1 \pm \beta)\Delta$.

Since every element $e \in Y_{\text{good}}$ satisfies $\deg_{\mathcal{J}}(e) = (1 \pm \beta)\Delta$, we assert that Y_{good} verifies Condition (2) in the hypothesis of Theorem 4.1. In particular, we assert that $|Y_{\text{good}}| \geq (1-\beta)|X|$. (If true, the proof of Lemma 2.7 will be complete.) To see the assertion, we have

$$|Y_{\text{good}}| = |Y| - |Y_{\text{bad}}| \overset{(58)}{\geq} (1 - (\beta/2))\, |X| - |Y_{\text{bad}}|.$$

Thus, our assertion is implied once we prove the following claim.

Claim 4.7. $|Y_{\text{bad}}| < (\beta/2)|X|$.

Proof of Claim 4.7. Double-counting gives that

$$\sum_{J \in \overline{\mathcal{J}}} |J \cap (W \setminus X)| = \sum_{e \in W \setminus X} \deg_{\tilde{\mathcal{J}}}(e).$$

By construction of $\overline{\mathcal{J}}$, each term in the former sum is at least one, and by Setup 2.5, each term of the latter sum is at most m^{f-3}. Thus, $|\overline{\mathcal{J}}| \leq |W \setminus X| \times m^{f-3}$, where the first part of (55) gives

$$|W \setminus X| \leq \mu \sum_{\{a,b,c\} \in \mathcal{F}_0} |\mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac})|.$$

As such,

$$|\overline{\mathcal{J}}| \leq \mu m^{f-3} \sum_{\{a,b,c\} \in \mathcal{F}_0} |\mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac})|. \tag{61}$$

On the other hand, double-counting also gives that

$$\sum_{J \in \overline{\mathcal{J}}} |J \cap Y_{\text{bad}}| = \sum_{e \in Y_{\text{bad}}} |\overline{\mathcal{J}}_e|.$$

Trivially, each term in the former sum is at most j (in fact, each term is at most $j-1$, since every $J \in \overline{\mathcal{J}}$ also overlaps $W \setminus X$), and by definition of Y_{bad}, each term in the latter sum is bounded from below according to (60). As such,

$$j \times |\overline{\mathcal{J}}| \geq |Y_{\text{bad}}| \times \zeta \alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} \geq |Y_{\text{bad}}| \times \zeta \alpha_0^{|\mathcal{F}_0|} d^{\binom{f}{2}} m^{f-3}. \tag{62}$$

Comparing (61) and (62), we see

$$\begin{aligned} |Y_{\text{bad}}| &\leq \mu j \zeta^{-1} \alpha_0^{-|\mathcal{F}_0|} d^{-\binom{f}{2}} \sum_{\{a,b,c\} \in \mathcal{F}_0} |\mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac})| \overset{(54)}{\leq} \mu j \zeta^{-1} \alpha_0^{-|\mathcal{F}_0|} d^{-\binom{f}{2}} (\alpha - \delta)^{-1} |W| \\ &\overset{(55)}{\leq} \mu j \zeta^{-1} \alpha_0^{-|\mathcal{F}_0|} d^{-\binom{f}{2}} (\alpha_0 - \delta)^{-1} (1 - (\rho/2))^{-1} |X| \overset{(49)}{<} (\beta/2)|X|, \end{aligned}$$

as desired. □

5. Proof of the Triple Extension Lemma (Lemma 4.3)

In this section, we prove Lemma 4.3 by deriving it from a 'Counting Lemma' of Haxell, Nagle and Rödl [9]. (This derivation resembles one of Cooley et al. [2].) For that, we consider the following setup which generalizes Setup 2.5, and which uses (for reasons seen in context) some alternative notation.

Setup 5.1 (Counting Setup)**.** Suppose $\alpha_0, \delta, d_0, \varepsilon > 0$ and $r, m \in \mathbb{N}$ are given together with a graph R and 3-graph $\mathcal{H}$ satisfying the following conditions:

(1) $R = \bigcup\{R^{ij} : 1 \leq i < j \leq r\}$ is an r-partite graph with vertex partition $V_1 \cup \cdots \cup V_r$, $|V_1| = \cdots = |V_r| = m$, where for each $1 \leq i < j \leq r$, $R^{ij} = R[V_i, V_j]$ is (d_{ij}, ε)-regular, for some $d_{ij} \geq d_0$;

(2) $\mathcal{H} = \bigcup\{\mathcal{H}^{hij} : 1 \leq h < i < j \leq r\} \subseteq \mathcal{K}_3(R)$ is a 3-graph satisfying that, for each $1 \leq h < i < j \leq r$, $\mathcal{H}^{hij} = \mathcal{H}[V_h, V_i, V_j]$ is (α_{hij}, δ)-minimal w.r.t. $R^{hi} \cup R^{ij} \cup R^{hj}$, for some $\alpha_{hij} \geq \alpha_0$, meaning (cf. Definition 2.3)

$$|\mathcal{K}^{(3)}_{2,2,2}(\mathcal{H}^{hij})| \leq \alpha^8_{hij}(d_{hi}d_{ij}d_{hj})^4 \binom{m}{2}^3 (1+\delta). \tag{63}$$

In the context of Setup 5.1, let $\mathcal{K}_r(\mathcal{H}) = \binom{\mathcal{H}}{K_r^{(3)}}$ denote the family of r-cliques in $\mathcal{H}$.

Theorem 5.2 (Counting Lemma [9])**.** *For every $r \in \mathbb{N}$ and for all $\alpha_0, \xi > 0$, there exists $\delta = \delta_{\text{Thm.5.2}}(r, \alpha_0, \xi) > 0$ so that, for all $d_0 > 0$, there exists $\varepsilon = \varepsilon_{\text{Thm.5.2}}(r, \alpha_0, \xi, \delta, d_0) > 0$ so that the following holds.*

Let R and $\mathcal{H}$ satisfy the hypothesis of Setup 5.1 with $r, \alpha_0, \delta, d_0, \varepsilon > 0$ above, and with a sufficiently large integer m. Then, $|\mathcal{K}_r(\mathcal{H})| = (1 \pm \xi) \prod_{1 \leq h < i < j \leq r} \alpha_{hij} \times \prod_{1 \leq h < i \leq r} d_{ij} \times m^r$.

Remark 5.3. In [9], Theorem 5.2 was proven in the case when all $\alpha_{hij} = \alpha_0$, $1 \leq h < i < j \leq r$, and all $d_{ij} = d_0$, $1 \leq h < i \leq r$. It is well-known and standard to show that this case implies the Counting Lemma in full. For completeness, we sketch these details in the Appendix (Section 7). □

To prove Lemma 4.3, we also use the following version of the Cauchy-Schwarz inequality.

Fact 5.4 (Cauchy-Schwarz Inequality)**.** *For $a_1, \ldots, a_t \geq 0$ and $\beta \geq 0$, suppose $\sum_{i=1}^t a_i \geq (1-\beta)at$ and $\sum_{i=1}^t a_i^2 \leq (1+\beta)a^2 t$. Then, all but $2\beta^{1/3}t$ indices $1 \leq i \leq t$ satisfy $a_i = a(1 \pm 2\beta^{1/3})$.*

We also use the following easy fact.

Fact 5.5. *For all $\delta, d_0 > 0$, there exists $\varepsilon = \varepsilon_{\text{Fact5.5}}(\delta, d_0) > 0$ so that the following holds. Let R satisfy the hypothesis of Setup 5.1 with $r = 3$, $d_0, \varepsilon > 0$ above, and m sufficiently large. Let $\mathcal{H} = \mathcal{K}_3(R)$. Then, $\mathcal{H}$ is necessarily $(1, \delta)$-minimal w.r.t. R.*

Proof. Indeed, it is well-known that for a suitable function $f(\varepsilon) \to 0$ as $\varepsilon \to 0$, the Graph Counting Lemma (see upcoming Lemma 7.3) implies that R contains within $(1 \pm f(\varepsilon))(d_{12}d_{23}d_{13})^4\binom{m}{2}^3$ many copies of the graph $K^{(2)}_{2,2,2}$. Thus, we take $\varepsilon > 0$ small enough so that $f(\varepsilon) < \delta$, and (63) is immediate. □

5.1. **Proof of Lemma 4.3.** Let $\mathcal{F}_0$ be a fixed 3-graph with $V(\mathcal{F}_0) = [f]$. Let $\alpha_0, \zeta > 0$ be given. To define the promised constant $\delta = \delta_{\text{Lem. 4.3}}(\mathcal{F}_0, \alpha_0, \zeta) > 0$, we first consider the following auxiliary parameters. Define

$$F = 2f - 3, \quad \text{and choose } \xi > 0 \text{ to satisfy} \quad 1 - \left(\frac{\zeta}{2}\right)^3 \leq (1-\xi)^{3+f^3} \leq (1+2\xi)^{2+2F^3} \leq 1 + \left(\frac{\zeta}{2}\right)^3. \tag{64}$$

With $r = f$, let $\delta_{\text{Thm. 5.2}}(f) = \delta_{\text{Thm. 5.2}}(f, \alpha_0, \xi) > 0$ be the constant guaranteed by Theorem 5.2. With $r = F$, let $\delta_{\text{Thm.5.2}}(F) = \delta_{\text{Thm.5.2}}(F, \alpha_0, \xi) > 0$ be the constant guaranteed by Theorem 5.2. Set

$$\delta = \delta_{\text{Lem.4.3}}(\mathcal{F}_0, \alpha_0, \zeta) = \min\{\alpha_0\xi,\ \delta_{\text{Thm.5.2}}(f),\ \delta_{\text{Thm.5.2}}(F)\}. \tag{65}$$

Let $d^{-1} \in \mathbb{N}$ be given. To define the promised constant $\varepsilon = \varepsilon_{\text{Lem. 4.3}}(\mathcal{F}_0, \alpha_0, \zeta, \delta, d) > 0$, we again consider some auxiliary parameters. With $d_0 = d$, let $\varepsilon_{\text{Fact5.5}}(\delta, d_0) > 0$ be the constant guaranteed by Fact 5.5. With $r = f$ and $d_0 = d$, let $\varepsilon_{\text{Thm.5.2}}(f) = \varepsilon_{\text{Thm.5.2}}(f, \alpha_0, \xi, \delta, d) > 0$ be the constant guaranteed by Theorem 5.2. With $r = F$ and $d_0 = d$, let $\varepsilon_{\mathrm{Thm.5.2}}(F) = \varepsilon_{\mathrm{Thm.5.2}}(F, \alpha_0, \xi, \delta, d) > 0$ be the constant guaranteed by Theorem 5.2. Finally, with $\tau = \xi$, let $\varepsilon_{\mathrm{Fact2.2}} = \varepsilon_{\mathrm{Fact2.2}}(d, \xi) > 0$ be the constant guaranteed by the Triangle Counting Lemma (Fact 2.2). Set

$$\varepsilon = \varepsilon_{\mathrm{Lem.4.3}}(\mathcal{F}_0, \alpha_0, \zeta, \delta, d) = \min\left\{\varepsilon_{\mathrm{Fact5.5}},\, \varepsilon_{\mathrm{Fact2.2}},\, \varepsilon_{\mathrm{Thm.5.2}}(f),\, \varepsilon_{\mathrm{Thm.5.2}}(F)\right\}. \tag{66}$$

Let P and $\mathcal{G}$ satisfy the hypothesis of Setup 2.5 with $\mathcal{F}_0$ and some $\alpha \geq \alpha_0$, with $\delta, d, \varepsilon > 0$ above, and with a sufficiently large integer m. To prove Lemma 4.3, we prove that for each fixed $\{h, i, j\} \in \mathcal{F}_0$, all but $\zeta|\mathcal{G}^{hij}|$ elements $\{u, v, w\} \in \mathcal{G}^{hij}$ satisfy

$$\mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{u, v, w\}) = (1 \pm \zeta)\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3}, \tag{67}$$

where $\mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{u, v, w\})$ denotes the number of *extensions* of $\{u, v, w\}$ to partite-isomorphic copies of $\mathcal{F}_0$ in $\mathcal{G}$. Without loss of generality, we assume $\{h, i, j\} = \{1, 2, 3\} \in \mathcal{F}_0$. The proof of (67) will use the Counting Lemma (Theorem 5.2) to show that

$$\sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{u, v, w\}) \geq \left(1 - \left(\frac{\zeta}{2}\right)^3\right) \left(\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3}\right) \times |\mathcal{G}^{123}|, \tag{68}$$

and

$$\sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}^2_{\mathcal{F}_0,\mathcal{G}}(\{u, v, w\}) \leq \left(1 + \left(\frac{\zeta}{2}\right)^3\right) \left(\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3}\right)^2 \times |\mathcal{G}^{123}|. \tag{69}$$

Using (68) and (69), the Cauchy-Schwarz Inequality (Fact 5.4) immediately concludes (67).

Proof of (68). Write

$$\#\left\{\mathcal{F}_0 \subseteq_{\mathrm{p.i.}} \mathcal{G}\right\} = \left|\left\{\mathcal{F} \in \binom{\mathcal{G}}{\mathcal{F}_0} : \mathcal{F} \text{ is a partite-isomorphic copy of } \mathcal{F}_0 \text{ in } \mathcal{G}\right\}\right|$$

for the number of partite-isomorphic copies of $\mathcal{F}_0$ in $\mathcal{G}$. Observe that $\sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{u, v, w\}) = \#\left\{\mathcal{F}_0 \subseteq_{\mathrm{p.i.}} \mathcal{G}\right\}$. To bound this quantity with the Counting Lemma, construct the following hypergraph $\mathcal{H}$ from $\mathcal{G}$: for $\{h, i, j\} \in \binom{[f]}{3}$, set

$$\mathcal{H}^{hij} = \begin{cases} \mathcal{G}^{hij} & \text{if } \{h, i, j\} \in \mathcal{F}_0, \\ \mathcal{K}_3(P^{hi} \cup P^{ij} \cup P^{hj}) & \text{if } \{h, i, j\} \notin \mathcal{F}_0 \end{cases}$$

and set $\mathcal{H} = \bigcup\left\{\mathcal{H}^{hij} : 1 \leq h < i < j \leq f\right\}$. Then, $\sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{u, v, w\}) = \#\left\{\mathcal{F}_0 \subseteq_{\mathrm{p.i.}} \mathcal{G}\right\} = |\mathcal{K}_f(\mathcal{H})|$. Since $\mathcal{G}$ and P satisfy the hypothesis of Setup 2.5, $\mathcal{H}$ and $R = P$ satisfy the hypothesis of Setup 5.1, specifically with $\alpha_{hij} = \alpha \pm \delta$ when $\{h, i, j\} \in \mathcal{F}_0$ and $\alpha_{hij} = 1$ otherwise, and all $d_{ij} = d$, $1 \leq i < j \leq f$ (cf. (65), (66) and Fact 5.5). An application of the Counting Lemma (Theorem 5.2) gives

$$\sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{u, v, w\}) \geq (1 - \xi) \prod_{1 \leq h < i < j \leq f} \alpha_{hij} \times \prod_{1 \leq i < j \leq f} d_{ij} \times m^f \geq (1 - \xi)(\alpha - \delta)^{|\mathcal{F}_0|} d^{\binom{f}{2}} m^f.$$

To infer (68), we rewrite the inequality above. By the Triangle Counting Lemma (Fact 2.2),

$$|\mathcal{G}^{123}| = \alpha_{123}|\mathcal{K}_3(P^{12} \cup P^{23} \cup P^{13})| = (\alpha \pm \delta)(1 \pm \xi) d^3 m^3. \tag{70}$$

As such (using $(1 + x)^{-1} \geq 1 - x$),

$$\sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}_{\mathcal{F}_0,\mathcal{G}}(\{u, v, w\}) \geq \frac{(1 - \xi)(\alpha - \delta)^{|\mathcal{F}_0|}}{(1 + \xi)(\alpha + \delta)} d^{\binom{f}{2}-3} m^{f-3} \times |\mathcal{G}^{123}|$$

$$\geq (1 - \xi)^2 \left(1 - \frac{\delta}{\alpha_0}\right)^{|\mathcal{F}_0|+1} \alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} \times |\mathcal{G}^{123}| \overset{(65)}{\geq} (1 - \xi)^{3+f^3} \alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3} \times |\mathcal{G}^{123}|$$

and so (68) follows from (64). □

Proof of (69). The proof is similar to its predecessor, but more involved. We begin by defining a graph R, and 3-graphs $\mathcal{F}_1$, $\mathcal{H}$ and $\hat{\mathcal{H}}$, and use the following notation: define $\phi : [F] \to [f]$ (cf. (64)) by

$$\phi(a) = \bar{a} = \begin{cases} a & \text{if } a \in [f], \\ a - f + 3 & \text{if } a \in [F] \setminus [f]. \end{cases}$$

For $a \in [F] \setminus [f]$, let V_a be a copy of $V_{\bar{a}}$ (where $V(P) = V(\mathcal{G}) = V_1 \cup \cdots \cup V_f$ from Setup 2.5).

Defining R. Set $V(R) = V_1 \cup \cdots \cup V_F$. For each $1 \le i < j \le f$, set $R^{ij} = P^{ij}$. For distinct $a \in \{1, 2, 3, f+1, \dots, F\}$ and $b \in \{f+1, \dots, F\}$, let R^{ab} be a copy of $P^{\bar{a}\bar{b}}$ (which is defined on $V_{\bar{a}} \cup V_{\bar{b}}$) defined on $V_a \cup V_b$. Finally, for $a \in \{4, \dots, f\}$ and $b \in \{f+1, \dots, F\}$, set $R^{ab} = K[V_a, V_b]$. (Any complete bipartite graph is $(1, 0)$-regular.) Set $R = \bigcup \left\{ R^{ab} : \{a, b\} \in \binom{[F]}{2} \right\}$. Note that $P \subseteq R$. By Setup 2.5 and the construction above, R has precisely $2\binom{f}{2} - 3$ many bipartite graphs R^{ab}, $1 \le a < b \le f$, which are (d, ε)-regular, and all remaining bipartite graphs R^{ab} of R are $(1, \varepsilon)$-regular.

Defining $\mathcal{F}_1$, $\mathcal{H}$ and $\hat{\mathcal{H}}$. Set $V(\mathcal{F}_1) = [F] = [2f - 3]$ (cf. (64)). Define

$$\mathcal{F}_0' = \left\{ \{a, b, c\} \in \binom{\{1, 2, 3, f+1, f+2, \dots, F\}}{3} : \{\bar{a}, \bar{b}, \bar{c}\} \in \mathcal{F}_0 \right\},$$

which is a copy of $\mathcal{F}_0$ on vertices $\{1, 2, 3, f+1, f+2, \dots, F\}$. Now, define $\mathcal{F}_1 = \mathcal{F}_0 \cup \mathcal{F}_0'$, which is the 3-graph defined on vertex set $[F]$ consisting of two copies of $\mathcal{F}_0$ (namely, $\mathcal{F}_0$ and $\mathcal{F}_0'$) which meet along the three shared vertices and single shared edge

$$V(\mathcal{F}_0) \cap V(\mathcal{F}_0') = \{1, 2, 3\} = \mathcal{F}_0 \cap \mathcal{F}_0'. \tag{71}$$

As such, $|\mathcal{F}_1| = 2|\mathcal{F}_0| - 1$. Set $V(\mathcal{H}) = V_1 \cup \cdots \cup V_F$. For each $\{h, i, j\} \in \mathcal{F}_0$, set $\mathcal{H}^{hij} = \mathcal{G}^{hij}$. For each $\{a, b, c\} \in \mathcal{F}_1 \setminus \mathcal{F}_0$, let $\mathcal{H}^{abc} \subseteq \mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac})$ be a copy of $\mathcal{G}^{\bar{a}\bar{b}\bar{c}} \subseteq \mathcal{K}_3(P^{\bar{a}\bar{b}} \cup P^{\bar{b}\bar{c}} \cup P^{\bar{a}\bar{c}})$. Finally, for each $\{a, b, c\} \in \binom{[F]}{3} \setminus \mathcal{F}_1$, set $\mathcal{H}^{abc} = \mathcal{K}_3(P^{ab} \cup P^{bc} \cup P^{ac})$. Set $\mathcal{H} = \bigcup \left\{ \mathcal{H}^{abc} : \{a, b, c\} \in \mathcal{F}_1 \right\}$ and $\hat{\mathcal{H}} = \bigcup \left\{ \mathcal{H}^{abc} : \{a, b, c\} \in \binom{[F]}{3} \right\}$. Note that $\mathcal{G} \subseteq \mathcal{H} \subseteq \hat{\mathcal{H}}$. By Setup 2.5 and the construction above, $\hat{\mathcal{H}}$ has precisely $|\mathcal{F}_1| = 2|\mathcal{F}_0| - 1$ many subhypergraphs $\mathcal{H}^{abc}$, $1 \le a < b < c \le F$, which are (α_{abc}, δ)-minimal, where $\alpha_{abc} = \alpha \pm \delta$, and all remaining subhypergraphs $\mathcal{H}^{abc}$ of $\hat{\mathcal{H}}$ are $(1, \delta)$-minimal (cf. Fact 5.5).

Observe that

$$\sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}^2_{\mathcal{F}_0, \mathcal{G}}(\{u, v, w\}) = \sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}_{\mathcal{F}_1, \mathcal{H}}(\{u, v, w\}).$$

Indeed, an F-tuple of vertices in $\mathcal{H}$ is a partite-isomorphic copy of $\mathcal{F}_1$ if, and only if, its restrictions to $\bigcup_{j \in [f]} V_j$ and $\bigcup_{j \in \{1, 2, 3, f+1, \dots, F\}} V_j$ yield partite-isomorphic copies of $\mathcal{F}_0$ and $\mathcal{F}_0'$, resp. As such,

$$\sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}^2_{\mathcal{F}_0, \mathcal{G}}(\{u, v, w\}) = \sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}_{\mathcal{F}_1, \mathcal{H}}(\{u, v, w\}) = \# \{\mathcal{F}_1 \subseteq_{\text{p.i.}} \mathcal{H}\} = |\mathcal{K}_F(\hat{\mathcal{H}})|.$$

Since $\mathcal{F}_0$, P and $\mathcal{G}$ satisfy the hypothesis of Setup 2.5, R and $\hat{\mathcal{H}}$ satisfy the hypothesis of Setup 5.1, specifically with $\alpha_{abc} = \alpha \pm \delta$ when $\{a, b, c\} \in \mathcal{F}_1$ and $\alpha_{abc} = 1$ otherwise (cf. Fact 5.5), and all $d_{ab} \in \{d, 1\}$, $1 \le a < b \le F$ (cf. (65), (66)). As such, an application of the Counting Lemma (Theorem 5.2) gives

$$\begin{aligned} \sum_{\{u,v,w\} \in \mathcal{G}^{123}} \mathrm{ext}^2_{\mathcal{F}_0, \mathcal{G}}(\{u, v, w\}) &\le (1 + \xi) \prod_{1 \le a < b < c \le F} \alpha_{abc} \times \prod_{1 \le a < b \le F} d_{ab} \times m^F \\ &\le (1 + \xi)(\alpha + \delta)^{|\mathcal{F}_1|} d^{2\binom{f}{2} - 3} m^{2f - 3} = (1 + \xi)(\alpha + \delta)^{2|\mathcal{F}_0| - 1} d^{2\binom{f}{2} - 3} m^{2f - 3} \\ &\overset{(70)}{\le} \frac{(1 + \xi)}{(1 - \xi)(\alpha - \delta)} (\alpha + \delta)^{2|\mathcal{F}_0| - 1} d^{2\binom{f}{2} - 6} m^{2f - 6} |\mathcal{G}^{123}|. \end{aligned}$$

Using $(1-x)^{-1} \leq 1+2x$ for $x \in [0,1/2]$, we further infer

$$\sum_{\{u,v,w\}\in\mathcal{G}^{123}} \mathrm{ext}^2_{\mathcal{F}_0,\mathcal{G}}(\{u,v,w\}) \leq (1+2\xi)^2 \left(1+2\frac{\delta}{\alpha_0}\right)^{2|\mathcal{F}_0|} \left(\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3}\right)^2 |\mathcal{G}^{123}|$$

$$\overset{(65)}{\leq} (1+2\xi)^{2+2F^3} \left(\alpha^{|\mathcal{F}_0|-1} d^{\binom{f}{2}-3} m^{f-3}\right)^2 |\mathcal{G}^{123}|,$$

and so (69) follows from (64). □

6. Proof of the Slicing Lemma (Lemma 2.4)

In our proof of Lemma 2.4, we use the following two lemmas.

Lemma 6.1 (Auxiliary Slicing Lemma)**.** *For all $\rho > 0$ and $s \in \mathbb{N}$, there exists $S_0 = S_{\mathrm{Lem.6.1}}(\rho, s) \in \mathbb{N}$ so that the following statement holds.*

Let $K^{(3)}[A,B,C]$ be the complete 3-partite 3-graph with vertex partition $A \cup B \cup C$, $|A| = |B| = |C| = S \geq S_0$. Let $q_1, \dots, q_s > 0$ be given where $\sum_{i=1}^s q_i \leq 1$. Then, there exists a partition $K^{(3)}[A,B,C] = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \dots \cup \mathcal{J}_s$ so that, for each $1 \leq i \leq s$, $|\mathcal{J}_i| = (q_i \pm \rho)S^3$ and $|\mathcal{K}_{2,2,2}(\mathcal{J}_i)| \leq q_i^8 \binom{S}{2}^3 (1+\rho)$.

Lemma 6.1 holds by the following standard probabilistic considerations. In the context above, independently for each $\{a,b,c\} \in K^{(3)}[A,B,C]$, place $\{a,b,c\} \in \mathcal{J}_i$ with probability q_i, $1 \leq i \leq s$, and place $\{a,b,c\} \in \mathcal{J}_0$ otherwise. To prove that the resulting partition has the desired properties, one appeals to the Chernoff and Janson inequalities (cf. [12]). For simplicity, we omit these details.

Lemma 6.2 (Inheritance Lemma)**.** *For all $\alpha_0, \delta_0 > 0$, there exists $\delta = \delta_{\mathrm{Lem.6.2}} > 0$ so that, for all $d > 0$, there exists $\varepsilon = \varepsilon_{\mathrm{Lem.6.2}} > 0$ so that the following statement holds.*

Let P be a triad (cf. Definition 2.1) with parameters $d, \varepsilon > 0$ above and sufficiently large integer m, and suppose $\mathcal{G} \subseteq \mathcal{K}_3(P)$ is (α, δ)-minimal w.r.t. P, for some $\alpha \geq \alpha_0$. Let $V_i' \subseteq V_i$, $1 \leq i \leq 3$, be given with $|V_1'| = |V_2'| = |V_3'| > \delta_0 m$. Then,

1. *for each $1 \leq i < j \leq 3$, $P[V_i', V_j']$ is $(d, \varepsilon/\delta_0)$-regular;*
2. *$\mathcal{G}[V_1', V_2', V_3']$ is (α', δ_0)-minimal w.r.t. $P[V_1', V_2', V_3']$ (cf. Definition 2.3), with $\alpha' = \alpha \pm \delta_0$.*

Statement (1) of Lemma 6.2 is well-known, but we prove Statement (2) in Section 6.2.

We shall also use the following easy fact. In what follows, $K^{(2)}_{2,2,2}$ denotes the complete 3-partite graph with 2 vertices in each vertex class. For a graph R, let $\mathcal{K}_{2,2,2}(R)$ denote the set of all copies of $K^{(2)}_{2,2,2}$ in R.

Fact 6.3. *For all $d > 0$, there exists $\varepsilon = \varepsilon_{\mathrm{Fact6.3}}(d) > 0$ so that the following statement holds. Suppose $R = R^{12} \cup R^{13} \cup R^{23}$ is a 3-partite graph with 3-partition $U_1 \cup U_2 \cup U_3$, where each R^{ij}, $1 \leq i < j \leq 3$, is (d, ε)-regular and where each $|U_i|$, $1 \leq i \leq 3$, is sufficiently large. Then, $|\mathcal{K}_{2,2,2}(R)| \leq 2d^{12}\binom{|U_1|}{2}\binom{|U_2|}{2}\binom{|U_3|}{2}$.*

Fact 6.3 is a standard and well-known consequence of the Graph Counting Lemma (Lemma 7.3 in the Appendix). We omit the proof.

6.1. **Proof of Lemma 2.4.** This proof involves quite a few constants, which we summarize below in (79). Let $\alpha_0, \delta' > 0$ be given. Let

$$0 < \rho < \frac{1}{16}\min\{\alpha_0, \delta'\} \quad \text{satisfy} \quad \left(1 + 16\frac{\rho}{\alpha_0}\right)^9 \leq 1 + \frac{\delta'}{2}. \tag{72}$$

With $\rho > 0$ given above, and for an integer $1 \leq s \leq 1/\alpha_0$, let $S_0(s) = S_{\mathrm{Lem.6.1}}(\rho, s)$ be the constant guaranteed by the Auxiliary Slicing Lemma (Lemma 6.1). Set $S_0 = \max_{1 \leq s \leq 1/\alpha_0} S_0(s)$, and set

$$S = \max\left\{S_0, \left\lceil \frac{2^9 \cdot 105}{\delta' \alpha_0^8} \right\rceil \right\}. \tag{73}$$

Set

$$3\tau = \xi = \frac{\rho}{3} \quad \text{and let} \quad \delta_{\text{Thm.5.2}} = \delta_{\text{Thm.5.2}}(r = 6, \alpha_0/2, \xi) \tag{74}$$

be the constant guaranteed by the Counting Lemma (Theorem 5.2). Set

$$0 < \delta_0 \le \min\left\{\frac{1}{2S}, \delta_{\text{Thm.5.2}}\right\} \quad \text{to satisfy} \quad \left(1 + \frac{\delta_0}{\alpha_0}\right)^8 \le 1 + \xi. \tag{75}$$

The Inheritance Lemma guarantees constant

$$\delta_{\text{Lem.6.2}} = \delta_{\text{Lem.6.2}}(\alpha_0, \delta_0) \quad \text{and set} \quad \delta = \delta_{\text{Lem.2.4}}(\alpha_0, \delta') = \delta_{\text{Lem.6.2}}. \tag{76}$$

Let $d > 0$ be given. Let

$$\varepsilon_{\text{Fact2.2}} = \varepsilon_{\text{Fact2.2}}(d, \tau), \quad \varepsilon_{\text{Fact5.5}} = \varepsilon_{\text{Fact5.5}}(\delta, d), \quad \varepsilon_{\text{Thm.5.2}} = \varepsilon_{\text{Thm.5.2}}(r = 6, \alpha_0, \xi, \delta_0, d),$$
$$\varepsilon_{\text{Lem.6.2}} = \varepsilon_{\text{Lem.6.2}}(\alpha_0, \delta_0, \delta, d), \quad \varepsilon_{\text{Fact6.3}} = \varepsilon_{\text{Fact6.3}}(d), \tag{77}$$

be the constants guaranteed by the Triangle Counting Lemma (Fact 2.2), Fact 5.5, the Counting Lemma (Theorem 5.2), the Inheritance Lemma (Lemma 6.2) and Fact 6.3, respectively. Set

$$\varepsilon = \varepsilon_{\text{Lem.2.4}}(\alpha_0, \delta', \delta, d) = \delta_0 \times \min\left\{\varepsilon_{\text{Fact2.2}}, \varepsilon_{\text{Fact5.5}}, \varepsilon_{\text{Thm.5.2}}, \varepsilon_{\text{Lem.6.2}}, \varepsilon_{\text{Fact6.3}}\right\}. \tag{78}$$

In all that follows, let m be a sufficiently large integer. The constants above can be summarized by the following hierarchies:

$$\alpha_0, \delta' \gg \rho > \xi > \tau;$$
$$\rho \gg \frac{1}{S}, \delta_{\text{Thm.5.2}} \gg \delta_0 \gg \delta_{\text{Lem.6.2}} = \delta;$$
$$d \gg \varepsilon_{\text{Fact2.2}}, \varepsilon_{\text{Fact5.5}}, \varepsilon_{\text{Thm.5.2}}, \varepsilon_{\text{Lem.6.2}}, \varepsilon_{\text{Fact6.3}} \gg \varepsilon \ggg \frac{1}{m}. \tag{79}$$

Let $P = P^{12} \cup P^{23} \cup P^{13}$ be a triad (cf. Definition 2.1) with parameters d, ε, m above, and let $\mathcal{G} \subseteq \mathcal{K}_3(P)$ be (α, δ)-minimal w.r.t. P, for some $\alpha \ge \alpha_0$. Let $\sigma_1, \dots, \sigma_s \ge \alpha_0$ be given with $\sum_{i=1}^s \sigma_i \le \alpha$. To define the partition promised by Lemma 2.4, we make two preparations.

First, we prepare an application of the Auxiliary Slicing Lemma (Lemma 6.1). Let $A = \{a_1, \dots, a_S\}$, $B = \{b_1, \dots, b_S\}$, $C = \{c_1, \dots, c_S\}$ be auxiliary sets (cf. (73)). Set $q_i = \sigma_i/\alpha > 0$, $1 \le i \le s$, so that

$$\sum_{i=1}^s \sigma_i \le \alpha \quad \Longrightarrow \quad \sum_{i=1}^s q_i \le 1. \tag{80}$$

For $q_1, \dots, q_s > 0$ defined above, let $K^{(3)}[A, B, C] = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \dots \cup \mathcal{J}_s$ be the partition guaranteed by Lemma 6.1. (This partition can be found by an exhaustive search in time depending on $S = O(1)$.) Second, we construct a vertex partition of $V(\mathcal{G}) = V(P)$ which 'mirrors' $A \cup B \cup C$. Fix $1 \le i \le 3$, and let

$$V_i = V_{i,0} \cup \bigcup_{x=1}^S V_{i,x} \tag{81}$$

be any partition satisfying $|V_{i,1}| = \dots = |V_{i,S}| = \lfloor m/S \rfloor$. (Clearly, such a partition is constructed in linear time $O(m)$.)

We now define the promised partition $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_s$. Fix $0 \le a, b, c \le S$. If $\{a, b, c\} \cap \{0\} \ne \emptyset$, put $\mathcal{G}[V_{1,a}, V_{2,b}, V_{3,c}] \subseteq \mathcal{G}_0$. Otherwise, for $0 \le k \le s$, put

$$\mathcal{G}[V_{1,a}, V_{2,b}, V_{3,c}] \subset \mathcal{G}_k \quad \Longleftrightarrow \quad \{a, b, c\} \in \mathcal{J}_k. \tag{82}$$

Since $s \le 1/\alpha_0$ and S (cf. (73)) are constants, the partition $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_s$ is constructed in time $O(m^3)$. It therefore remains to show that it has the desired property. For that, and for the remainder of the proof, fix $1 \le k \le s$. We show that $\mathcal{G}_k$ is (α_k, δ')-minimal w.r.t. P, where $d_{\mathcal{G}_k}(P) \stackrel{\text{def}}{=} \alpha_k = \sigma_k \pm \delta'$.

It is fairly easy to see that $d_{\mathcal{G}_k}(P) = \alpha_k = \sigma_k \pm \delta'$. Indeed,

$$|\mathcal{G}_k| = \sum_{\{a,b,c\}\in\mathcal{J}_k} |\mathcal{G}[V_{1,a}, V_{2,b}, V_{3,c}]| = \sum_{\{a,b,c\}\in\mathcal{J}_k} \alpha_{abc}|\mathcal{K}_3(P[V_{1,a}, V_{2,b}, V_{3,c}])|,$$

where we wrote, for each $\{a, b, c\} \in \mathcal{J}_k$, $\alpha_{abc} = |\mathcal{G}[V_{1,a}, V_{2,b}, V_{3,c}]|/|\mathcal{K}_3(P[V_{1,a}, V_{2,b}, V_{3,c}])|$. Now, for each fixed $\{a, b, c\} \in \mathcal{J}_k$, we will apply the Inheritance Lemma (Lemma 6.2). Since $|V_{1,a}| = |V_{2,b}| = |V_{3,c}| = \lfloor m/S \rfloor > \delta_0 m$ (cf. (75)), Lemma 6.2 (cf. (76)–(78)) guarantees that $\alpha_{abc} = \alpha \pm \delta_0$ and that each of $P[V_{1,a}, V_{2,b}]$, $P[V_{1,a}, V_{3,c}]$, $P[V_{2,b}, V_{3,c}]$ is $(d, \varepsilon/\delta_0)$-regular. Applying the Triangle Counting Lemma (Fact 2.2) (cf. (77), (78)), we have

$$|\mathcal{G}_k| = (\alpha \pm \delta_0) \sum_{\{a,b,c\}\in\mathcal{J}_k} |\mathcal{K}_3(P[V_{1,a}, V_{2,b}, V_{3,c}])| = (\alpha \pm \delta_0)(1 \pm \tau)d^3 \left\lfloor \frac{m}{S} \right\rfloor^3 |\mathcal{J}_k|.$$

Applying Lemma 6.1 (cf. (73)) and recalling $\alpha q_k = \sigma_k$, we see

$$|\mathcal{G}_k| = (1 \pm o(1))(q_k \pm \rho)(\alpha \pm \delta_0)(1 \pm \tau)d^3m^3 \overset{(79)}{=} (\sigma_k \pm 7\rho)\, d^3m^3.$$

Finally, by the Triangle Counting Lemma (Fact 2.2) (cf. (77), (78)), we see[2]

$$\alpha_k \overset{\text{def}}{=} d_{\mathcal{G}_k}(P) = \frac{|\mathcal{G}_k|}{|\mathcal{K}_3(P)|} = \frac{(\sigma_k \pm 7\rho)\, d^3m^3}{(1 \pm \tau)d^3m^3} \overset{(74)}{=} \sigma_k \pm 8\rho \overset{(72)}{=} \sigma_k \pm \delta', \tag{83}$$

as promised.

It remains to show that $\mathcal{G}_k$ is (α_k, δ')-minimal w.r.t. P, i.e.,

$$|\mathcal{K}_{2,2,2}(\mathcal{G}_k)| \le \alpha_k^8 d^{12} \binom{m}{2}^3 (1 + \delta'), \tag{84}$$

for which we establish some notation and terminology. Let $\mathcal{O} = \{a, a', b, b', c, c'\} \in \mathcal{K}_{2,2,2}(\mathcal{J}_k)$ be an *octohedron* of $\mathcal{J}_k$, where we understand $a, a' \in A$, $b, b' \in B$, $c, c' \in C$. Define $\mathcal{G}_{\mathcal{O}} = \mathcal{G}[V_{1,a} \cup V_{1,a'}, V_{2,b} \cup V_{2,b'}, V_{3,c} \cup V_{3,c'}]$ to be the subhypergraph of $\mathcal{G}$ induced on $V_{1,a} \cup V_{1,a'} \cup V_{2,b} \cup V_{2,b'} \cup V_{3,c} \cup V_{3,c'}$. Note that, since $\mathcal{O} \subseteq \mathcal{J}_k$, we have $\mathcal{G}_{\mathcal{O}} \subseteq \mathcal{G}_k$. We call an element $\hat{\mathcal{O}} = \{v_{1,a}, v_{1,a'}, v_{2,b}, v_{2,b'}, v_{3,c}, v_{3,c'}\} \in \mathcal{K}_{2,2,2}(\mathcal{G}_{\mathcal{O}})$ *standard* if

$$v_{1,a} \in V_{1,a}, \quad v_{1,a'} \in V_{1,a'}, \quad v_{2,b} \in V_{2,b}, \quad v_{2,b'} \in V_{2,b'}, \quad v_{3,c} \in V_{3,c}, \quad v_{3,c'} \in V_{3,c'}, \tag{85}$$

where, necessarily, $a \ne a'$, $b \ne b'$, $c \ne c'$. We write

$$\mathcal{K}^+_{2,2,2}(\mathcal{G}_{\mathcal{O}}) = \left\{\hat{\mathcal{O}} \in \mathcal{K}_{2,2,2}(\mathcal{G}_{\mathcal{O}}) : \hat{\mathcal{O}} \text{ standard}\right\}, \quad \mathcal{K}^+_{2,2,2}(\mathcal{G}_k) = \bigcup \left\{\mathcal{K}^+_{2,2,2}(\mathcal{G}_{\mathcal{O}}) : \mathcal{O} \in \mathcal{K}_{2,2,2}(\mathcal{J}_k)\right\},$$
$$\text{and} \quad \mathcal{K}^-_{2,2,2}(\mathcal{G}_k) = \mathcal{K}_{2,2,2}(\mathcal{G}_k) \setminus \mathcal{K}^+_{2,2,2}(\mathcal{G}_k).$$

We proceed with the following claims.

Claim 6.4. *For each* $\mathcal{O} \in \mathcal{K}_{2,2,2}(\mathcal{J}_k)$, $|\mathcal{K}^+_{2,2,2}(\mathcal{G}_{\mathcal{O}})| \le \alpha^8 d^{12} \lfloor m/S \rfloor^6 (1 + \rho)$.

Claim 6.5. $\left|\mathcal{K}^-_{2,2,2}(\mathcal{G}_k)\right| \le \frac{13}{S} d^{12}m^6$.

We postpone the proofs of Claims 6.4 and 6.5 in favor of finishing the proof of (84).

By Claims 6.4 and 6.5,

$$\begin{aligned} |\mathcal{K}_{2,2,2}(\mathcal{G}_k)| &= \left|\mathcal{K}^-_{2,2,2}(\mathcal{G}_k)\right| + \left|\mathcal{K}^+_{2,2,2}(\mathcal{G}_k)\right| = \left|\mathcal{K}^-_{2,2,2}(\mathcal{G}_k)\right| + \sum \left\{\left|\mathcal{K}^+_{2,2,2}(\mathcal{G}_{\mathcal{O}})\right| : \mathcal{O} \in \mathcal{K}_{2,2,2}(\mathcal{J}_k)\right\} \\ &\le \frac{13}{S} d^{12}m^6 + |\mathcal{K}_{2,2,2}(\mathcal{J}_k)| \times \alpha^8 d^{12} \lfloor m/S \rfloor^6 (1 + \rho)\,. \end{aligned}$$

---

[2] We write

$$x = \frac{a \pm b}{c \pm d} \iff \frac{a - b}{c + d} \le x \le \frac{a + b}{c - d}.$$

Applying Lemma 6.1 (cf. (73)) and recalling $\alpha q_k = \sigma_k \leq \alpha_k + 8\rho$ (cf. (83)), we see

$$|\mathcal{K}_{2,2,2}(\mathcal{G}_k)| \leq (q_k\alpha)^8 d^{12}\frac{m^6}{8}(1+\rho)^2 + \frac{13}{S}d^{12}m^6 \leq \left((1+o(1))\sigma_k^8(1+3\rho) + \frac{105}{S}\right)d^{12}\binom{m}{2}^3$$
$$\stackrel{(83)}{\leq} \left((\alpha_k+8\rho)^8(1+4\rho) + \frac{105}{S}\right)d^{12}\binom{m}{2}^3 \leq \left(\left(1+8\frac{\rho}{\alpha_k}\right)^9 + \frac{105}{S\alpha_k^8}\right)\alpha_k^8 d^{12}\binom{m}{2}^3.$$

From (83), we have that $\alpha_k \geq \sigma_k - 8\rho \geq \alpha_0 - 8\rho \stackrel{(72)}{\geq} \alpha_0/2$ (recall, by hypothesis, $\sigma_k \geq \alpha_0$). Therefore,

$$|\mathcal{K}_{2,2,2}(\mathcal{G}_k)| \leq \left(\left(1+16\frac{\rho}{\alpha_0}\right)^9 + 2^8\frac{105}{S\alpha_0^8}\right)\alpha_k^8 d^{12}\binom{m}{2}^3 \stackrel{(72),(73)}{\leq} (1+\delta')\,\alpha_k^8 d^{12}\binom{m}{2}^3,$$

which proves (84).

Proof of Claim 6.4. Fix $\mathcal{O} = \{a,a',b,b',c,c'\} \in \mathcal{K}_{2,2,2}(\mathcal{J}_k)$. We apply the Counting Lemma (Theorem 5.2) to $\mathcal{G}_{\mathcal{O}}$, in the following way. Define a 6-partition $V_{1,a}\cup V_{1,a'}\cup V_{2,b}\cup V_{2,b'}\cup V_{3,c}\cup V_{3,c'}$, where $|V_{1,a}| = |V_{1,a'}| = |V_{2,b}| = |V_{2,b'}| = |V_{3,c}| = |V_{3,c'}| = \left\lfloor\frac{m}{S}\right\rfloor$, which will be the vertex set of the following 6-partite graph R and 3-graph $\mathcal{H} \subseteq \mathcal{K}_3(R)$. For each $(a_0,b_0,c_0) \in \{a,a'\}\times\{b,b'\}\times\{c,c'\}$, define $R^{a_0b_0} = P[V_{1,a_0},V_{2,b_0}]$, $R^{b_0c_0} = P[V_{2,b_0},V_{3,c_0}]$, $R^{a_0c_0} = P[V_{1,a_0},V_{3,c_0}]$. By the Inheritance Lemma (Lemma 6.2) (cf. (77), (78)), each of these 12 bipartite graphs is $(d,2S\varepsilon)$-regular. Define $R^{a,a'} = K[V_{1,a},V_{1,a'}]$, $R^{bb'} = K[V_{2,b},V_{2,b'}]$, $R^{cc'} = K[V_{3,c},V_{3,c'}]$. Each of these bipartite graphs is $(1,o(1))$-regular. Set

$$R = \bigcup\left\{R^{xy} : \{x,y\} \in \binom{\{a,a',b,b',c,c'\}}{2}\right\}. \tag{86}$$

For each $(a_0,b_0,c_0) \in \{a,a'\}\times\{b,b'\}\times\{c,c'\}$, define $\mathcal{H}^{a_0b_0c_0} = \mathcal{G}[V_{1,a_0},V_{2,b_0},V_{3,c_0}]$. By the Inheritance Lemma (Lemma 6.2) (cf. (76)–(78)), we have that $\mathcal{H}^{a_0b_0c_0}$ is $(\alpha_{a_0b_0c_0},\delta_0)$-minimal with respect to $R[V_{1,a_0},V_{2,b_0},V_{3,c_0}]$, for some $\alpha_{a_0b_0c_0} = \alpha \pm \delta_0$. We now define 12 further 3-partite 3-graphs. To that end, fix $\{x,y,z\} \in \binom{\{a,a',b,b',c,c'\}}{3}$ where, for some $d \in \{a,b,c\}$, we have $\{d,d'\} \subset \{x,y,z\}$. Define $\mathcal{H}^{xyz} = \mathcal{K}_3(R^{xy}\cup R^{yz}\cup R^{xy})$ so that, by Fact 5.5, (cf. (77), (78)) $\mathcal{H}^{xyz}$ is $(1,\delta)$-minimal w.r.t. $R^{xy}\cup R^{yz}\cup R^{xz}$. Set

$$\mathcal{H} = \bigcup\left\{\mathcal{H}^{xyz} : \{x,y,z\} \in \binom{\{a,a',b,b',c,c'\}}{3}\right\}. \tag{87}$$

By the construction (86) and (87), every clique K_6 in $\mathcal{H}$ corresponds to a copy $\hat{\mathcal{O}} \in \mathcal{K}^+_{2,2,2}(\mathcal{G}_{\mathcal{O}})$, and vice-versa. Moreover, by construction, R and $\mathcal{H}$ satisfy the hypothesis of Theorem 5.2 with $r=6$, α_0, δ_0, d, $2S\varepsilon$ (cf. (75)–(78)). Thus, by the Counting Lemma (recall $\alpha \geq \alpha_0$),

$$|\mathcal{K}^+_{2,2,2}(\mathcal{G}_{\mathcal{O}})| = |\mathcal{K}_6(\mathcal{H})| \leq (1+\xi)\prod\left\{\alpha_{a_0b_0c_0} : (a_0,b_0,c_0) \in \{a,a'\}\times\{b,b'\}\times\{c,c'\}\right\}\times d^{12}\left\lfloor\frac{m}{S}\right\rfloor^6$$
$$\leq (1+\xi)(\alpha+\delta_0)^8 d^{12}\left\lfloor\frac{m}{S}\right\rfloor^6 \leq (1+\xi)\left(1+\frac{\delta_0}{\alpha_0}\right)^8\alpha^8 d^{12}\left\lfloor\frac{m}{S}\right\rfloor^6 \stackrel{(75)}{\leq} (1+\xi)^2\alpha^8 d^{12}\left\lfloor\frac{m}{S}\right\rfloor^6,$$

and so Claim 6.4 follows from (74). □

Proof of Claim 6.5. Indeed, note that

$$\hat{\mathcal{O}} = \{v_{1,a},v_{1,a'},v_{2,b},v_{2,b'},v_{3,c},v_{3,c'}\} \in \mathcal{K}^-_{2,2,2}(\mathcal{G}_k) \iff$$
$$\text{either } \hat{\mathcal{O}}\cap V_{i,0} \neq \emptyset \text{ for some } i\in[3]\text{, or (85) holds with } a=a' \text{ or } b=b' \text{ or } c=c'.$$

Clearly, at most $3Sm^5 = O(m^5)$ elements $\hat{\mathcal{O}} \in \mathcal{K}_{2,2,2}(\mathcal{G}_k)$ can satisfy $\hat{\mathcal{O}}\cap(V_{1,0}\cup V_{2,0}\cup V_{3,0}) \neq \emptyset$. To enumerate $\hat{\mathcal{O}} \in \mathcal{K}^-_{2,2,2}(\mathcal{G}_k)$ of the latter variety, fix $1 \leq a,a',b,b',c,c' \leq S$ where, w.l.o.g., $a=a'$. We use Fact 6.3 to estimate

$$|\mathcal{K}_{2,2,2}(\mathcal{G}_k[V_{1,a},V_{2,b}\cup V_{2,b'},V_{3,c}\cup V_{3,c'}])| \leq |\mathcal{K}_{2,2,2}(P[V_{1,a},V_{2,b}\cup V_{2,b'},V_{3,c}\cup V_{3,c'}])|\,.$$

To apply Fact 6.3, set $U_1 = V_{1,a}$, $U_2 = V_{2,b} \cup V_{2,b'}$, $U_3 = V_{3,c} \cup V_{3,c'}$ and $R^{ij} = P[U_i, U_j]$ for all $1 \le i < j \le 3$. Since each $|U_i| \ge \lfloor m/S \rfloor$, $1 \le i \le s$, the Inheritance Lemma (Lemma 6.2) (cf. (77), (78)) guarantees that each R^{ij}, $1 \le i < j \le 3$, is $(d, 2S\varepsilon)$-regular. Fact 6.3 (cf. (77), (78)) then guarantees

$$|\mathcal{K}_{2,2,2}(P[V_{1,a}, V_{2,b} \cup V_{2,b'}, V_{3,c} \cup V_{3,c'}])| \le 2d^{12} \binom{|V_{1,a}|}{2} \binom{|V_{2,b} \cup V_{2,b'}|}{2} \binom{|V_{3,c} \cup V_{3,c'}|}{2} \le 4d^{12} \frac{m^6}{S^6}.$$

Summing over all $3S^5$ many 5-element indices $1 \le a, a', b, b', c, c' \le S$, we conclude

$$|\mathcal{K}^-_{2,2,2}(\mathcal{G}_k)| \le O(m^5) + 12S^5 d^{12} \frac{m^6}{S^6} \le \frac{13}{S} d^{12} m^6,$$

as promised. □

6.2. **Proof of Lemma 6.2.** We prove Statement (2) of the Inheritance Lemma, and use the following important concept of Frankl and Rödl [6].

Definition 6.6 ((α, δ)-regularity)**.** Let P be a triad (cf. Definition 2.1) and let $\mathcal{G} \subseteq \mathcal{K}_3(P)$ satisfy $d_{\mathcal{G}}(P) = \alpha$. For $\delta > 0$, we say that $\mathcal{G}$ is (α, δ)-*regular* if, for all $Q \subseteq P$ with $|\mathcal{K}_3(Q)| > \delta |\mathcal{K}_3(P)|$, we have $|d_{\mathcal{G}}(Q) - \alpha| < \delta$.

It was shown by Nagle, Poerschke, Rödl and Schacht (see Theorem 2.1 and Corollary 2.1 in [14]) that, with suitably defined constants, (α, δ)-minimality and (α, δ)-regularity are equivalent concepts.

Theorem 6.7 (Nagle, Poerschke, Rödl, Schacht [14])**.** *For all $\alpha_0, \hat{\delta} > 0$, there exists $\delta = \delta_{\mathrm{Thm.6.7}}(\alpha_0, \hat{\delta}) > 0$ so that, for all $d > 0$, there exists $\varepsilon = \varepsilon_{\mathrm{Thm.6.7}}(\alpha_0, \hat{\delta}, \delta, d) > 0$ so that the following statement holds.*

Let P be a triad with parameters d, ε and a sufficiently large integer m, and let $\mathcal{G} \subseteq \mathcal{K}_3(P)$ satisfy $d_{\mathcal{G}}(P) = \alpha \ge \alpha_0$.

1. *If $\mathcal{G}$ is (α, δ)-minimal w.r.t. P, then $\mathcal{G}$ is $(\alpha, \hat{\delta})$-regular w.r.t. P.*
2. *If $\mathcal{G}$ is (α, δ)-regular w.r.t. P, then $\mathcal{G}$ is $(\alpha, \hat{\delta})$-minimal w.r.t. P.*

By using Theorem 6.7, the proof of Lemma 6.2 is a formality.

Proof of Lemma 6.2. Let $\alpha_0, \delta_0 > 0$ be given. With $\hat{\delta} = \delta_0$, Theorem 6.7 ensures constant

$$\delta_1 = \delta_{\mathrm{Thm.6.7}}(\alpha_0 - \delta_0, \delta_0) > 0, \quad \text{and set} \quad \delta_2 = \frac{1}{4} \delta_0^3 \delta_1. \tag{88}$$

With $\hat{\delta} = \delta_2$, Theorem 6.7 ensures constant

$$\delta_3 = \delta_{\mathrm{Thm.6.7}}(\alpha_0, \delta_2) > 0, \quad \text{and set} \quad \delta = \delta_{\mathrm{Lem.6.2}}(\alpha_0, \delta_0) = \delta_3. \tag{89}$$

Let $d > 0$ be given. With $\tau = 1/2$, let

$$\varepsilon_0 = \varepsilon_{\mathrm{Fact2.2}}(d, \tau = 1/2) > 0 \tag{90}$$

be the constant guaranteed by the Triangle Counting Lemma (Fact 2.2). With $\hat{\delta} = \delta_0$, let

$$\varepsilon_1 = \varepsilon_{\mathrm{Thm6.7}}(\alpha_0 - \delta_0, \delta_0, \delta_1, d) > 0 \tag{91}$$

be the constant guaranteed by Theorem 6.7. With $\hat{\delta} = \delta_2$, let

$$\varepsilon_2 = \varepsilon_{\mathrm{Thm6.7}}(\alpha_0, \delta_2, \delta, d) > 0 \tag{92}$$

be the constant guaranteed by Theorem 6.7. Set

$$\varepsilon = \varepsilon_{\mathrm{Lem.6.2}}(\alpha_0, \delta_0, \delta, d) = \min\{\delta_0 \varepsilon_0, \delta_0 \varepsilon_1, \varepsilon_2\}. \tag{93}$$

In all that follows, let m be a sufficiently large integer.

Let P be a triad (cf. Definition 2.1) with parameters d, ε, m above, and suppose $\mathcal{G} \subseteq \mathcal{K}_3(P)$ is (α, δ)-minimal w.r.t. P. Let $V_i' \subseteq V_i$, $1 \le i \le 3$, be given with $|V_1'| = |V_2'| = |V_3'| > \delta_0 m$. To simplify notation in the argument below, we write $P' = P[V_1', V_2', V_3']$, $\mathcal{G}' = \mathcal{G} \cap \mathcal{K}_3(P')$, $\alpha' = d_{\mathcal{G}'}(P')$. By Statement (1) of Lemma 6.2, each bipartite graph $P[V_i', V_j']$, $1 \le i < j \le 3$, is $(d, \varepsilon/\delta_0)$-regular, and so by (93), each such $P[V_i', V_j']$ is (d, ε_0)-regular and (d, ε_1)-regular.

We first apply Theorem 6.7 to $\mathcal{G}$ and P. To that end, recall from our hypothesis that $\mathcal{G}$ is (α, δ)-minimal w.r.t. P, where $\alpha \geq \alpha_0$ and where each P^{ij}, $1 \leq i < j \leq 3$, is (d, ε)-regular (cf. (89), (92) and (93)). As such, Theorem 6.7 ensures that $\mathcal{G}$ is (α, δ_2)-regular w.r.t. P. We proceed with the following claim.

Claim 6.8. *$\mathcal{G}'$ is (α', δ_1)-regular w.r.t. P', where $\alpha' = \alpha \pm \delta_2$.*

Proof of Claim 6.8. We first check that $\alpha' = \alpha \pm \delta_2$. Using Statement (1) of Lemma 6.2, it follows from (two applications of) the Triangle Counting Lemma (Fact 2.2) that

$$|\mathcal{K}_3(P')| > \frac{1}{2} d^3 |V_1'||V_2'||V_3'| > \frac{1}{2} \delta_0^3 d^3 |V_1||V_2||V_3| > \frac{1}{4} \delta_0^3 |\mathcal{K}_3(P)| \overset{(88)}{>} \delta_2 |\mathcal{K}_3(P)|. \tag{94}$$

Setting $Q = P'$ in Definition 6.6, we conclude from the (α, δ_2)-regularity of $\mathcal{G}$ w.r.t. P that $|\alpha' - \alpha| < \delta_2$, as promised.

Now, let $Q \subseteq \mathcal{K}_3(P')$ be given satisfying $|\mathcal{K}_3(Q)| > \delta_1 |\mathcal{K}_3(P')| \overset{(88)}{=} (4\delta_2/\delta_0^3)|\mathcal{K}_3(P')|$. By the penultimate bound of (94), we see $|\mathcal{K}_3(Q)| > \delta_2 |\mathcal{K}_3(P)|$, and so by the (α, δ_2)-regularity of $\mathcal{G}$ w.r.t. P, we have $|d_{\mathcal{G}}(Q) - \alpha| < \delta_2$, where clearly, $d_{\mathcal{G}}(Q) = d_{\mathcal{G}'}(Q)$. By the Triangle Inequality, $|d_{\mathcal{G}'}(Q) - \alpha'| \leq 2\delta_2 \overset{(88)}{<} \delta_1$, concluding the proof. □

We now apply Theorem 6.7 to $\mathcal{G}'$ and P'. To that end, we have from Claim 6.8 that $\mathcal{G}'$ is (α', δ_1)-regular w.r.t. P', where $\alpha' \geq \alpha - \delta_2 \geq \alpha_0 - \delta_0$, and where each constituent bipartite graph $P[V_i', V_j']$, $1 \leq i < j \leq 3$, is (d, ε_1)-regular (cf. (88), (91), and (93)). As such, Theorem 6.7 ensures that $\mathcal{G}'$ is (α', δ_0)-minimal w.r.t. P', as promised. □

7. Appendix: Proof of Theorem 5.2

As we mentioned in Remark 5.3, the Counting Lemma was proven in [9] in the following special case.

Theorem 7.1 (Haxell, Nagle, Rödl [9])**.** *For every $r \in \mathbb{N}$ and for all $\alpha, \mu > 0$, there exists $\delta = \delta_{\text{Thm. 7.1}}(r, \alpha, \mu) > 0$ so that, for all $d > 0$, there exists $\varepsilon = \varepsilon_{\text{Thm. 7.1}}(r, \alpha, \mu, \delta, d) > 0$ so that the following holds.*

Let $R = P$ and $\mathcal{H} = \mathcal{G}$ satisfy the hypothesis of Setup 5.1 with $r, \alpha_0 = \alpha, \delta, d_0 = d, \varepsilon > 0$ above, where all $\alpha_{hij} = \alpha$, $1 \leq h < i < j \leq r$, and all $d_{ij} = d$, $1 \leq i < j \leq r$, and where $m \in \mathbb{N}$ is sufficiently large. Then, $|\mathcal{K}_r(\mathcal{G})| = (1 \pm \mu) \alpha^{\binom{r}{3}} d^{\binom{r}{2}} m^r$.

By adapting the proof in [9] of Theorem 7.1 (in only symbolic ways), one would arrive at Theorem 5.2. For completeness, we follow lines from [15, 16] to sketch a proof that, in fact, Theorem 5.2 can be derived from Theorem 7.1. We begin by making the following remark about how we use Theorem 7.1.

Remark 7.2. We use that Theorem 7.1 holds under the slightly more general hypothesis that, for all $1 \leq h < i < j \leq r$, we have $\alpha_{hij} = \alpha + o(1)$, where $o(1) \to 0$ as $m \to \infty$. This statement holds by following the proof in [9] verbatim, and including factors of $1 + o(1)$ whenever needed. □

For our proof-sketch, we use two very well-known results referenced earlier in this paper: the *Graph Counting Lemma* and the *Graph Extension Lemma*. We state these results without proof, and use the following notation. For a graph R satisfying the hypothesis of Setup 5.1, set $\mathcal{K}_r(R) = \binom{R}{K_r^{(2)}}$, and for each $\{u, v, w\} \in \mathcal{K}_3(R)$, set $\text{ext}_{K_r^{(2)}, R}(\{u, v, w\}) = |\{U \in \mathcal{K}_r(R) : \{u, v, w\} \subseteq U\}|$.

Lemma 7.3 (Graph Counting and Extension Lemma)**.** *For every $r \in \mathbb{N}$ and for all $d_0, \tau > 0$, there exists $\varepsilon = \varepsilon_{\text{Lem.7.3}}(r, d_0, \tau) > 0$ so that the following holds.*

Let R be a graph satisfying the hypothesis of Setup 5.1 with $r, d_0, \varepsilon > 0$ above, and where $m \in \mathbb{N}$ is sufficiently large. Then,

(1) $|\mathcal{K}_r(R)| = (1 \pm \tau) \prod_{1 \leq i < j \leq r} d_{ij} \times m^r$;

(2) *for each* $1 \le h < i < j \le r$, *all but* τm^3 *triangles* $\{v_h, v_i, v_j\} \in \mathcal{K}_3(R^{hi} \cup R^{ij} \cup R^{hj})$ *satisfy*

$$\mathrm{ext}_{K_r^{(2)},R}(\{v_h, v_i, v_j\}) = (1 \pm \tau)\frac{1}{d_{hi}d_{ij}d_{hj}} \prod_{1 \le a < b \le r} d_{ab} \times m^{r-3}.$$

7.1. **Proof-sketch of Theorem 5.2.** We omit a formal description of constants and refer to the hierarchy $1/r, \alpha_0, \xi \gg \delta \gg d_0 \gg \varepsilon \ggg 1/m$, which is consistent with the quantification of Theorem 5.2. In the hierarchy above, we introduce further constants μ, α, d, τ (cf. Remark 7.5):

$$\frac{1}{r}, \alpha_0, \xi \gg \mu, \alpha \gg \delta \gg d_0 \gg d, \tau \gg \varepsilon \ggg \frac{1}{m}, \tag{95}$$

which is consistent with the quantification of Theorem 7.1. Now, let R and $\mathcal{H}$ satisfy the hypothesis of Setup 5.1 with the constants $\alpha_0, \delta, d_0, \varepsilon, m$. We show

$$|\mathcal{K}_r(\mathcal{H})| = (1 \pm \xi) \prod_{1 \le h < i < j \le r} \alpha_{hij} \times \prod_{1 \le i < j \le r} d_{ij} \times m^r. \tag{96}$$

The main idea for proving (96) is to 'randomly' partition the graph R and 3-graph $\mathcal{H}$ in a standard but suitable way. Fix $1 \le h < i < j \le r$, and set $p_{hij} = \alpha/\alpha_{hij}$ (cf. (95)) and $A_{hij} = \lfloor 1/p_{hij} \rfloor$. Define $\mathcal{H}^{hij} = \mathcal{H}_0^{hij} \cup \mathcal{H}_1^{hij} \cup \cdots \cup \mathcal{H}_{A_{hij}}^{hij}$ by the following rule: independently for each $\{v_h, v_i, v_j\} \in \mathcal{H}^{hij}$, put $\{v_h, v_i, v_j\} \in \mathcal{H}_a^{hij}$, $1 \le a \le A_{hij}$, with probability p_{hij}, and put $\{v_h, v_i, v_j\} \in \mathcal{H}_0^{hij}$ otherwise. We define a random partition of R similarly. Fix $1 \le i < j \le r$, and set $q_{ij} = d/d_R(V_i, V_j)$ and $B_{ij} = \lfloor 1/q_{ij} \rfloor$. Define $R^{ij} = R_0^{ij} \cup R_1^{ij} \cup \cdots \cup R_{B_{ij}}^{ij}$ by the following rule: independently for each $\{v_i, v_j\} \in R^{ij}$, put $\{v_i, v_j\} \in R_b^{ij}$, $1 \le b \le B_{ij}$, with probability q_{ij}, and put $\{v_i, v_j\} \in R_0^{ij}$ otherwise. We set

$$\mathcal{H}_0 = \bigcup \left\{ \mathcal{H}_0^{hij} : 1 \le h < i < j \le r \right\} \quad \text{and} \quad R_0 = \bigcup \left\{ R_0^{ij} : 1 \le i < j \le r \right\}. \tag{97}$$

We have the following standard probabilistic fact.

Fact 7.4. *With probability* $1 - o(1)$*, where* $o(1) \to 0$ *as* $m \to \infty$*, the following hold:*

(1) *for all* $1 \le i < j \le r$ *and* $1 \le b \le B_{ij}$*, the graph* R_b^{ij} *is* $(d, 2\varepsilon)$*-regular;*
(2) *for all* $\{h, i, j\} \in \binom{[r]}{3}$ *and* $(a, b_{hi}, b_{ij}, b_{hj}) \in [A_{hij}] \times [B_{hi}] \times [B_{ij}] \times [B_{hj}]$*, the 3-graph* $\mathcal{H}_a^{hij} \cap \mathcal{K}_3(P_{b_{hi}}^{hi} \cup P_{b_{ij}}^{ij} \cup P_{b_{hj}}^{hj})$ *is* $(\alpha \pm o(1), 2\delta)$*-minimal w.r.t.* $P_{b_{hi}}^{hi} \cup P_{b_{ij}}^{ij} \cup P_{b_{hj}}^{hj}$.

With probability at least 1/6, the following hold:

(3) *the graph* R_0 *satisfies* $|R_0| \le 3 \sum_{1 \le i < j \le r} q_{ij} |R^{ij}| = 3\binom{r}{2} d m^2$.
(4) *for each* $\{h, i, j\} \in \binom{[r]}{3}$, $|\mathcal{H}_0^{hij}| \le 2\binom{r}{3} p_{hij} |\mathcal{H}^{hij}| = 2\binom{r}{3}\alpha |\mathcal{K}_3(P^{hi} \cup P^{ij} \cup P^{hj})| \le 4r^3 \alpha d_{hi} d_{ij} d_{hj} m^3$.

We omit the proof of Fact 7.4, but indicate its key ingredients. Statement (1) is a well-known and routine application of the Chernoff inequality. Statement (2) is a routine application of the Janson inequality. Statements (3) and (4) are immediate applications of the Markov inequality. The last inequality in Statement (4) follows from the Graph Counting Lemma (Lemma 7.3) with $r = 3$.

We may now argue (96). We call a function $\boldsymbol{a} : \binom{[r]}{3} \to \mathbb{N}$ an $\mathcal{H}$*-pattern* if, for every $1 \le h < i < j \le r$, we have $\boldsymbol{a}(\{h, i, j\}) \stackrel{\text{def}}{=} \boldsymbol{a}_{hij} \le A_{hij}$. We call a function $\boldsymbol{b} : \binom{[r]}{2} \to \mathbb{N}$ an R*-pattern* if, for every $1 \le i < j \le r$, we have $\boldsymbol{b}(\{i, j\}) \stackrel{\text{def}}{=} \boldsymbol{b}_{ij} \le B_{ij}$. We call a pair of functions $(\boldsymbol{a}, \boldsymbol{b})$ an $(\mathcal{H}, R)$*-pattern* if $\boldsymbol{a}$ is an $\mathcal{H}$-pattern and $\boldsymbol{b}$ is an R-pattern. For an $(\mathcal{H}, R)$-pattern $(\boldsymbol{a}, \boldsymbol{b})$, we write

$$\mathcal{H}_{\boldsymbol{a}} = \bigcup \left\{ \mathcal{H}_{\boldsymbol{a}_{hij}}^{hij} : \{h, i, j\} \in \binom{[r]}{3} \right\}, \quad R_{\boldsymbol{b}} = \bigcup \left\{ R_{\boldsymbol{b}_{ij}}^{ij} : \{i, j\} \in \binom{[r]}{2} \right\}, \quad \mathcal{H}_{\boldsymbol{ab}} = \mathcal{H}_{\boldsymbol{a}} \cap \mathcal{K}_3(R_{\boldsymbol{b}}).$$

Since there are (exactly)

$$\prod_{1 \le h < i < j \le r} A_{hij} \times \prod_{1 \le i < j \le r} B_{ij} = \prod_{1 \le h < i < j \le r} \left\lfloor \frac{\alpha_{hij}}{\alpha} \right\rfloor \times \prod_{1 \le i < j \le r} \left\lfloor \frac{d_R(V_i, V_j)}{d} \right\rfloor \le \alpha^{-\binom{r}{3}} d^{-\binom{r}{2}} = O(1) \tag{98}$$

many $(\mathcal{H}, R)$-patterns $(\boldsymbol{a}, \boldsymbol{b})$, Fact 7.4 implies that, with high probability, every one of them yields a pair $\mathcal{G} = \mathcal{H}_{\boldsymbol{ab}}$ and $P = R_{\boldsymbol{b}}$ which satisfies the hypothesis of Theorem 7.1 with the constants $r, \alpha \pm o(1), 2\delta, d, 2\varepsilon, m$ (cf. (95)). As such, we may immediately conclude the lower bound in (96):

$$\begin{aligned}|\mathcal{K}_r(\mathcal{H})| &\geq \sum \{|\mathcal{K}_r(\mathcal{H}_{\boldsymbol{ab}})| : (\boldsymbol{a}, \boldsymbol{b}) \text{ is an } (\mathcal{H}, R)\text{-pattern}\} \\ &\overset{\text{Thm.7.1}}{\geq} (1-\mu)\alpha^{\binom{r}{3}} d^{\binom{r}{2}} m^r \times \prod_{1 \leq h < i < j \leq r} A_{hij} \times \prod_{1 \leq i < j \leq r} B_{ij} \\ &\overset{(98)}{\geq} (1-\mu) \prod_{1 \leq h < i < j \leq r} (\alpha_{hij} - \alpha) \times \prod_{1 \leq i < j \leq r} (d_R(V_i, V_j) - d) \times m^r \\ &\overset{(95)}{\geq} (1-\xi) \prod_{1 \leq h < i < j \leq r} \alpha_{hij} \times \prod_{1 \leq i < j \leq r} d_{ij} \times m^r,\end{aligned}$$

where we used $d_R(V_i, V_j) \geq d_{ij} - \varepsilon$, $1 \leq i < j \leq r$. It remains to prove the upper bound in (96).

Observe that

$$\begin{aligned}|\mathcal{K}_r(\mathcal{H})| \leq &\sum \{|\mathcal{K}_r(\mathcal{H}_{\boldsymbol{ab}})| : (\boldsymbol{a}, \boldsymbol{b}) \text{ is an } (\mathcal{H}, R)\text{-pattern}\} \\ &+ \left|\left\{\mathcal{R} \in \mathcal{K}_r(\mathcal{H}) : \binom{\mathcal{R}}{2} \cap R_0 \neq \emptyset\right\}\right| + \left|\left\{\mathcal{R} \in \mathcal{K}_r(\mathcal{H}) : \binom{\mathcal{R}}{3} \cap \mathcal{H}_0 \neq \emptyset\right\}\right|. \quad (99)\end{aligned}$$

An application of Theorem 7.1 (cf. (95)) yields

$$\begin{aligned}\sum \{|\mathcal{K}_r(\mathcal{H}_{\boldsymbol{ab}})| : (\boldsymbol{a}, \boldsymbol{b}) \text{ is an } (\mathcal{H}, R)\text{-pattern}\} &\leq (1+\mu)\alpha^{\binom{r}{3}} d^{\binom{r}{2}} m^r \times \prod_{1 \leq h < i < j \leq r} A_{hij} \times \prod_{1 \leq i < j \leq r} B_{ij} \\ &\overset{(98)}{\leq} (1+\mu) \prod_{1 \leq h < i < j \leq r} \alpha_{hij} \times \prod_{1 \leq i < j \leq r} d_{ij} \times m^r. \quad (100)\end{aligned}$$

Clearly,

$$\left|\left\{\mathcal{R} \in \mathcal{K}_r(\mathcal{H}) : \binom{\mathcal{R}}{2} \cap R_0 \neq \emptyset\right\}\right| \leq |R_0| m^{r-2} \overset{\text{Fact7.4}}{\leq} 3r^2 d m^r. \quad (101)$$

Momentarily, we will prove the following bound (see Remark 7.5):

$$\left|\left\{\mathcal{R} \in \mathcal{K}_r(\mathcal{H}) : \binom{\mathcal{R}}{3} \cap \mathcal{H}_0 \neq \emptyset\right\}\right| \leq r^3 \tau m^r + 8r^6 \alpha \prod_{1 \leq i < j \leq r} d_{ij} \times m^r. \quad (102)$$

Combining (99)–(102) and applying $\alpha_{hij} \geq \alpha_0$ and $d_{ij} \geq d_0$, $\{h, i, j\} \in \binom{[r]}{3}$, we have

$$\begin{aligned}|\mathcal{K}_r(\mathcal{H})| &\leq \prod_{1 \leq h < i < j \leq r} \alpha_{hij} \times \prod_{1 \leq i < j \leq r} d_{ij} \times m^r \left(1 + \mu + 3r^2 d \alpha_0^{-\binom{r}{3}} d_0^{-\binom{r}{2}} + r^3 \tau \alpha_0^{-\binom{r}{3}} d_0^{-\binom{r}{2}} + 8r^6 \alpha \alpha_0^{-\binom{r}{3}}\right) \\ &\overset{(95)}{\leq} (1+\xi) \prod_{1 \leq h < i < j \leq r} \alpha_{hij} \times \prod_{1 \leq i < j \leq r} d_{ij} \times m^r, \quad (103)\end{aligned}$$

as promised.

Remark 7.5. In (100), (102) and (103), the constants in (95) play a subtle role. We introduced the constant α in (95) to satisfy $\delta \ll \alpha \ll \alpha_0$. Indeed, we used take $\alpha \ll \alpha_0$ in (103), but we used $\delta \ll \alpha$ in (100) (for applying Theorem 7.1). The quantification of Theorem 5.2 allows $d_0 \ll \delta$, so we must accept $\alpha \gg d_0$, which makes (102) necessarily more delicate than (101). □

Proof of (102). Note that

$$\left|\left\{\mathcal{R} \in \mathcal{K}_r(\mathcal{H}) : \binom{\mathcal{R}}{3} \cap \mathcal{H}_0 \neq \emptyset\right\}\right| \leq \sum_{1 \leq h < i < j \leq r} \left|\left\{\mathcal{R} \in \mathcal{K}_r(\mathcal{H}) : \binom{\mathcal{R}}{3} \cap \mathcal{H}_0^{hij} \neq \emptyset\right\}\right|. \quad (104)$$

Now, fix $1 \leq h < i < j \leq r$. Clearly

$$\left|\left\{\mathcal{R} \in \mathcal{K}_r(\mathcal{H}) : \binom{\mathcal{R}}{3} \cap \mathcal{H}_0^{hij} \neq \emptyset\right\}\right| = \sum_{\{v_h, v_i, v_j\} \in \mathcal{H}_0^{hij}} \mathrm{ext}_{K_r^{(3)}, \mathcal{H}}(\{v_h, v_i, v_j\})$$

$$\leq \sum_{\{v_h, v_i, v_j\} \in \mathcal{H}_0^{hij}} \mathrm{ext}_{K_r^{(2)}, R}(\{v_h, v_i, v_j\}) \overset{\text{Lem.7.3}}{\leq} \tau m^r + 2\frac{1}{d_{hi} d_{ij} d_{hj}} \prod_{1 \leq a < b \leq r} d_{ab} \times m^{r-3} \times |\mathcal{H}_0^{hij}|$$

$$\overset{\text{Fact7.4}}{\leq} \tau m^r + 8r^3 \alpha \prod_{1 \leq a < b \leq r} d_{ab} \times m^r. \quad (105)$$

Applying (105) to (104) yields (102). □

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