ASYMPTOTICS OF THE EXTREMAL EXCEDANCE SET STATISTIC

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Abstract. For non-negative integers $r+s=n-1$, let $[b^r a^s]$ denote the number of permutations $\pi \in S_n$ which have the property that $\pi(i) > i$ if, and only if, $i \in [r] = \{1, \ldots, r\}$. Answering a question of Clark and Ehrenborg, we determine asymptotics on $[b^r a^s]$ when $r = \lfloor (n-1)/2 \rfloor$:

$$\left[b^{\lfloor (n-1)/2 \rfloor} a^{\lceil (n-1)/2 \rceil}\right] = \left(\frac{1}{2 \log 2 \sqrt{(1 - \log 2)}} + o(1) \right) \left(\frac{1}{2 \log 2} \right)^n n!.$$

We also determine asymptotics on $[b^r a^s]$ for a suitably related $r = \Theta(s) \to \infty$. Our proof depends on multivariate asymptotic methods of R. Pemantle and M. C. Wilson.

We also consider two applications of our main result. One, we determine asymptotics on the number of permutations $\pi \in S_n$ which simultaneously avoid the generalized patterns 21-34 and 34-21, i.e., for which π is order-isomorphic to neither (2,1,3,4) nor $(3,4,2,1)$ on any coordinates $1 \le i < i+1 < j < j+1 \le n$. We also determine asymptotics on the number of n-cycles $\pi \in C_n$ which avoid stretching pairs, i.e., those for which $1 \le \pi(i) < i < j < \pi(j) \le n$.

Keywords: excedance, descent, generalized patterns, combinatorial dynamics

1. Introduction

The main results of this paper (Theorems 1.2 and 1.3 below) consider the asymptotics of the number of permutations $\pi \in S_n$ with excedance word $b^r a^s$, where $r+s=n-1$. To present our results, we use the following notation and terminology, taken mostly from [4].

1.1. **Preliminaries.** Let S_n denote the set of permutations on $[n] = \{1, \ldots, n\}$. For $\pi \in S_n$ and $i \in [n-1]$, we say that i is an *excedance* of π if $\pi(i) > i$. We write $E(\pi) = \{i \in [n-1] : \pi(i) > i\}$ for the set of excedances of π. We define the *excedance word* $w(\pi) = (w_1, \ldots, w_{n-1}) \in \{a, b\}^{n-1}$ of π by, for each $i \in [n-1]$, $w_i = b$ if $i \in E(\pi)$ and $w_i = a$ if $i \notin E(\pi)$, i.e., $\pi(i) \le i$. For a word $w \in \{a, b\}^{n-1}$, let $[w]$ denote the number of permutations $\pi \in S_n$ for which $w(\pi) = w$.

In this paper, we consider words $w \in \{a, b\}^{n-1}$ of the following form. For non-negative integers $r+s=n-1$, let $b^r a^s \in \{a, b\}^{n-1}$ denote the word whose first r coordinates equal b and whose last s coordinates equal a. R. Ehrenborg and E. Steingrímsson [9] showed that $[b^r a^s]$ is maximized when $r = n-1-s \in \{\lfloor (n-1)/2 \rfloor, \lceil (n-1)/2 \rceil\}$. E. Clark and R. Ehrenborg [4, Concluding Remarks] posed the following problem.

Question 1.1. *What are the asymptotics of* $[b^{\lfloor (n-1)/2 \rfloor} a^{\lceil (n-1)/2 \rceil}]$ *as* $n \to \infty$*?*

In this paper, we address Question 1.1 (see Theorem 1.2 below), and more generally, in Theorem 1.3 we address the asymptotics of $[b^r a^s]$ for a wider range of $r+s=n-1$. We also provide some further applications of our main results.

1.2. **Historical remark.** An important counterpart of the notion of an excedance is that of a *descent*. For $\pi \in S_n$ and $i \in [n-1]$, we say that i is an *descent* of π if $\pi(i) > \pi(i+1)$. Two of the classically studied permutation statistics are the number of descents and the number of excedances (both considered by MacMahon [14]). The more modern study of permutation statistics has included the dual consideration of the number of permutations with prescribed descent set (and, respectively, excedance set). In particular, Niven [15] and, indepedently, de Bruijn [7] showed that the most common descent set is realized by the

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alternating permutations. The alternating permutations are enumerated by the Euler numbers E_n whose asymptotics are classical:

$$E_n = \left(\frac{4}{\pi} + o(1)\right)\left(\frac{2}{\pi}\right)^n n! \tag{1.1}$$

(This result follows from André's Theorem [2].) The most common excedance set was specified by the above-mentioned result [9], and the excedance set counterpart of (1.1) is provided by Theorem 1.2 below.

1.3. **Main Results.** Our first result provides an answer to Question 1.1.

Theorem 1.2.

$$\left[b^{\lfloor (n-1)/2 \rfloor} a^{\lceil (n-1)/2 \rceil}\right] = \left(\frac{1}{2\log 2\sqrt{(1-\log 2)}} + o(1)\right)\left(\frac{1}{2\log 2}\right)^n n!.$$

We prove Theorem 1.2 in Section 3.

More generally, we establish bivariate asymptotics for $[b^{r-1}a^s]$ when the parameters r and s tend to infinity in a suitable way. Our result is somewhat technical to present, and we require several considerations. To begin, define

$$\varepsilon_0 = (e-1)(1-\log(e-1)) \approx 0.7881... \quad \text{and} \quad S_{\varepsilon_0} = \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \varepsilon_0 \le \frac{s}{r} \le \frac{1}{\varepsilon_0}\right\}, \tag{1.2}$$

where we call S_{ε_0} an ε_0-*sector*. Our main result will provide asymptotics for $[b^{r-1}a^s]$ for $r, s \to \infty$ with $(r,s) \in S_{\varepsilon_0}$. To present our result, we require further considerations.

Define the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f(t) = \frac{(1-e^t)\log(1-e^{-t})}{t}, \quad t > 0. \tag{1.3}$$

We prove in the Appendix (cf. Fact 1) that f is invertible. Now, for $(r,s) \in S_{\varepsilon_0}$, write $x = x(r,s) := f^{-1}(s/r)$ and $y = y(r,s) := f^{-1}(r/s)$, and define the function $Q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$Q(x,y) = xye^{-x-y}\left[ye^{-y} + xe^{-x} - xy(e^{-y} + e^{-x})\right], \quad x, y > 0. \tag{1.4}$$

Our main result is now stated as follows.

Theorem 1.3. *For $(r,s) \in S_{\varepsilon_0}$, let $x = x(r,s)$, $y = y(r,s)$, and $Q(x,y)$ be defined as above. Then, the following holds uniformly for $r, s \to \infty$ with $(r,s) \in S_{\varepsilon_0}$:*

$$[b^{r-1}a^s] = s!r!\left(e^{-y} + O(s^{-1/2})\right)\frac{1}{\sqrt{2\pi}}x^{-r}y^{-s}\sqrt{\frac{ye^{-y}}{sQ(x,y)}}.$$

Theorem 1.3 is proved in Section 2 using a result of R. Pemantle and M.C. Wilson [17, 18, 19] concerning multivariate asymptotics.

1.4. **Applications.** We discuss two applications of Theorem 1.2. The first falls within the well-studied area of *generalized pattern avoidance*. The second concerns objects related to generalized patterns, which have become known as *stretching pairs*. Initially, it will not be obvious how Theorem 1.2 provides the applications (corollaries) we state, but we will make this connection clear at the end of the Introduction.

1.4.1. *Generalized patterns.* For $1 \le k \le n$, fix $\pi_0 \in S_k$ and $\pi \in S_n$. We say that π contains π_0 as a *pattern* if there exist $1 \le \ell_1 < \cdots < \ell_k \le n$ on which π is order-isomorphic to π_0. Now, write $\pi_0 = (a_1, \ldots, a_k) = (\pi_0(1), \ldots, \pi_0(k))$, and let $\pi_0^* = (a_1, \varepsilon_1, a_2, \varepsilon_2, \ldots, \varepsilon_{k-1}, a_k)$ be any sequence where, for each $1 \le i \le k$, ε_i is either a dash '$-$', or the empty string. We say that $\pi \in S_n$ admits π_0^* as a *generalized pattern* if it contains a pattern $1 \le \ell_1 < \cdots < \ell_k \le n$ of π_0 where, for all $1 \le i \le k$, $\varepsilon_i \ne -$ implies $\ell_{i+1} = \ell_i + 1$, i.e., ℓ_i, ℓ_{i+1} are consecutive in π.

For $\pi_0 \in S_k$ and π_0^* as above, write $\alpha_n(\pi_0^*)$ for the number of permutations $\pi \in S_n$ avoiding π_0^* as a generalized pattern. Elizalde and Noy [11] studied the limiting behavior of $(\alpha_n(\pi_0^*)/n!)^{1/n}$ for several 'dashless' (consecutive) generalized patterns π_0^* of length 3. For example, they showed that for

$\pi_0^* = 123 = (1,2,3)$ (with no dashes), one has $(\alpha_n(123)/n!)^{1/n} \to 3\sqrt{3}/(2\pi)$ as $n \to \infty$. Elizalde [10] showed that for $\pi_0^* = 1\text{-}23\text{-}4 = (1,-,2,3,-,4)$, one has $(\alpha_n(1\text{-}23\text{-}4)/n!)^{1/n} \to 0$ as $n \to \infty$. In [6], it was shown that for all sufficiently large even integers n, $(\alpha_n(21\text{-}34)/n!)^{1/n} \geq 1/2 + o(1)$, which answered a question of S. Elizalde [10]. Using Theorem 1.2, we are able to improve this last result by a factor of $1/\log 2$. To that end, let $\alpha_n(\{21\text{-}34, 34\text{-}21\})$ denote the number of permutations avoiding both 21-34 and 34-21 as generalized patterns.

Corollary 1.4.
$$\lim_{n\to\infty} \left(\frac{\alpha_n(\{21\text{-}34, 34\text{-}21\})}{n!} \right)^{1/n} = \frac{1}{2\log 2} \approx 0.7213...$$
As such,
$$\alpha_n(21\text{-}34) \geq \alpha_n(\{21\text{-}34, 34\text{-}21\}) = \left(\frac{1}{2\log 2} + o(1) \right)^n n!.$$

We prove Corollary 1.4 at the end of this section. It is an open problem to determine the exact value of the following limit (assuming that it exists):
$$\lim_{n\to\infty} \left(\frac{\alpha_n(21\text{-}34)}{n!} \right)^{1/n}.$$

1.4.2. *Stretching pairs.* For a permutation $\pi \in S_n$, we call a pair $1 \leq i < j \leq n$ a *stretching pair* if $\pi(i) < i < j < \pi(j)$. Stretching pairs naturally arise in cyclic permutations within the context of a well-known result of Sharkovsky [20] in discrete dynamical systems. More recently, stretching pairs in n-cycles were studied in [6, 13, 16] from a combinatorial point of view. Let $C_n \subset S_n$ denote the set of n-cycles, and let C_n^* denote those n-cycles which contain no stretching pairs. Theorem 1.2 allows us to determine the limiting behavior of $(|C_{n+1}^*|/n!)^{1/n}$.

Corollary 1.5.
$$\lim_{n\to\infty} \left(\frac{|C_{n+1}^*|}{n!} \right)^{1/n} = \frac{1}{2\log 2}.$$

We now proceed to the proofs of Corollaries 1.4 and 1.5.

1.5. **Proofs of Corollaries 1.4 and 1.5.** The following lemma allows us to connect Theorem 1.2 to Corollaries 1.4 and 1.5. As we show in Section 4, Lemma 1.6 is (mostly) a consequence of a result of Clarke, Steingrímsson, and Zeng [5].

Lemma 1.6.
(1) $|C_{n+1}^*| = \sum_{k=0}^{n-1} [b^k a^{n-1-k}]$.
(2) $|C_{n+1}^*| \leq \alpha_n(\{21\text{-}34, 34\text{-}21\}) \leq |C_{n+1}^*| + |C_{n+2}^*| \leq 2|C_{n+2}^*|$.

We now conclude Corollaries 1.4 and 1.5. Indeed, recall that is was proven in [9] that $[b^k a^{n-1-k}]$ is maximized when $k \in \{\lfloor (n-1)/2 \rfloor, \lceil (n-1)/2 \rceil\}$. As such, we may use Statement (1) of Lemma 1.6 to infer
$$\left[b^{\lfloor (n-1)/2 \rfloor} a^{\lceil (n-1)/2 \rceil} \right] \leq |C_{n+1}^*| \leq n \left[b^{\lfloor (n-1)/2 \rfloor} a^{\lceil (n-1)/2 \rceil} \right]$$
so that Corollary 1.5 is now immediate. Using Corollary 1.5 and Statement (2) of Lemma 1.6, Corollary 1.4 is now immediate.

Remark 1. Statement (1) of Lemma 1.6 also provides an exact combinatorial evaluation of $|C_{n+1}^*|$ that may be of independent interest. Indeed, Clark and Ehrenborg [4] showed that for all integers $M, N \geq 0$,
$$[b^M a^N] = \sum_{i \geq 0} \big(S(M+1, i+1) S(N+1, i+1) i!(i+1)! \big),$$

where $S(s,t)$, $s,t \in \mathbb{N}$, denotes the Stirling number of the second kind. As such,

$$|C^*_{n+1}| = \sum_{k=0}^{n-1} \sum_{i \geq 0} \big(S(k+1,i+1)S(n-k,i+1)i!(i+1)!\big),$$

which complements our result in Corollary 1.5.

2. Proof of Theorem 1.3

We begin by outlining the main approach to our proof of Theorem 1.3. To that end, we note a result of Clark and Ehrenborg (see [4, Thm. 3.1]) that $[b^r a^s]$ has bivariate exponential generating function

$$\sum_{r,s \geq 0} \frac{[b^r a^s]}{r!s!} x^r y^s = \frac{e^{-x}e^{-y}}{(e^{-x}+e^{-y}-1)^2} = \frac{\partial}{\partial x}\left(\frac{e^{-y}}{e^{-x}+e^{-y}-1}\right). \tag{2.1}$$

Now, for (x,y) in a neighborhood of $(0,0)$, write

$$\frac{e^{-y}}{e^{-x}+e^{-y}-1} = \sum_{r,s \geq 0} A_{r,s} x^r y^s. \tag{2.2}$$

We will apply a result of Pemantle and Wilson [17, 18, 19] to the coefficients $A_{r,s}$ when $(r,s) \in S_{\varepsilon_0}$ (cf. (1.2)), which will provide asymptotics on these coefficients. From there, Theorem 1.3 will follow, since by (2.1) we have

$$[b^{r-1}a^s] = r!s!A_{r,s}. \tag{2.3}$$

While the plan above is straightforward, the details take some work. Indeed, the result of Pemantle and Wilson is a highly technical statement, and much of our work will be in showing that it can be applied to the setting we need. Let us now proceed to the result of Pemantle and Wilson.

2.1. **Preliminaries and the result of Pemantle and Wilson.** In all that follows, suppose $F : \mathbb{C}^2 \to \mathbb{C}$ is a meromorphic function, where we write $F(x,y) = G(x,y)/H(x,y)$ for some holomorphic functions $G, H : \mathbb{C}^2 \to \mathbb{C}$. We write $\mathcal{V} = \mathcal{V}_F = \big\{(x,y) \in \mathbb{C}^2 : H(x,y) = 0\big\}$ for the variety of singularities of F. We say that H *vanishes to order one* on $\mathcal{V}$ if $\nabla H(x,y) \neq \vec{0}$ for each $(x,y) \in \mathcal{V}$. We write $\mathbf{dir}(x,y) = \mathrm{span}_{\mathbb{C}}\{(xH_x, yH_y)\}$.

On the variety $\mathcal{V}$, we have the following important concept of a *strictly minimal point*.

Definition 1. A point $(x,y) \in \mathcal{V}$ is called *strictly minimal* if the closed bidisk

$$\overline{D}(0,|x|) \times \overline{D}(0,|y|) = \{(z,w) \in \mathbb{C}^2 : |z| \leq |x|, |w| \leq |y|\}$$

intersects $\mathcal{V}$ only at the point (x,y).

The following result of Pemantle and Wilson appeared as Theorem 3.1 in [17], as Corollary 3.21 in [18], and as Theorem 9.5.7 in [19].

Theorem 2.1 (Pemantle, Wilson [17, 18, 19])**.** *Let $F = G/H : \mathbb{C}^2 \to \mathbb{C}$ be a meromorphic function with variety of singularities $\mathcal{V} = \mathcal{V}_F$. Write $F(x,y) = \sum_{r,s\geq 0} A_{r,s}x^r y^s$ outside of $\mathcal{V}$, and assume H vanishes to order one on $\mathcal{V}$.*

Fix $\varepsilon > 0$, and assume that for each $(r,s) \in S_\varepsilon$ (cf. (1.2)), there exists $(x,y) \in \mathcal{V}$ so that the following conditions hold:

(*i*) *$(r,s) \in \mathbf{dir}(x,y)$, where the point $(x,y) \in \mathcal{V}$ is strictly minimal;*
(*ii*) *As $(r,s) \in S_\varepsilon$ varies, the point $(x,y) \in \mathcal{V}$ varies smoothly over some compact set;*
(*iii*) *The point $(x,y) \in \mathcal{V}$ satisfies that $G(x,y) \neq 0$, and also,*

$$Q(x,y) := -y^2H_y^2 xH_x - yH_y x^2 H_x^2 - x^2y^2(H_y^2 H_{xx} + H_x^2 H_{yy} - 2H_xH_yH_{xy}) \neq 0.$$

Then as $r, s \to \infty$ with $(r,s) \in S_\varepsilon$, we have

$$A_{r,s} = \Big(G(x,y) + O(s^{-1/2})\Big) \frac{1}{\sqrt{2\pi}} x^{-r} y^{-s} \sqrt{\frac{-yH_y}{sQ(x,y)}}, \tag{2.4}$$

where the error estimate $O(s^{-1/2})$ is uniform over S_ε.

2.2. **Deriving Theorem 1.3 from Theorem 2.1.** From (2.1), set

$$F(x,y) = \frac{e^{-y}}{e^{-x} + e^{-y} - 1}, \quad G(x,y) = e^{-y}, \quad H(x,y) = e^{-x} + e^{-y} - 1,$$
$$\text{and} \quad \mathcal{V} = \mathcal{V}_F = \left\{(x,y) \in \mathbb{C}^2 : H(x,y) = e^{-x} + e^{-y} - 1 = 0\right\}. \tag{2.5}$$

Clearly, G and H are holomorphic functions so that F is meromorphic with variety of singularities given by $\mathcal{V}$. Clearly, ∇H is never zero, and therefore H vanishes to order one on $\mathcal{V}$.

Fix $(r,s) \in S_{\varepsilon_0}$ (cf. (1.2)). To apply Theorem 2.1, we must guarantee a point $(x,y) \in \mathcal{V}$ so that Conditions (i)–(iii) above hold. To find the desired point $(x,y) \in \mathcal{V}$, consider the equation

$$r\frac{e^x}{x} = s\frac{e^y}{y} \quad \text{on the variety } \mathcal{V}. \tag{2.6}$$

Using the identity $y = -\log(1 - e^{-x})$ on $\mathcal{V}$, we see that (2.6) holds if, and only if,

$$y = \frac{s}{r} \cdot \frac{xe^y}{e^x} = \frac{s}{r} \cdot \frac{x}{e^x - 1}.$$

Again using $y = -\log(1 - e^{-x})$ on $\mathcal{V}$, we see that (2.6) holds for x satisfying

$$-\log\left(1 - e^{-x}\right) = \frac{s}{r} \cdot \frac{x}{e^x - 1} \quad \Longrightarrow \quad \frac{s}{r} = \frac{(1 - e^x)\log(1 - e^{-x})}{x}. \tag{2.7}$$

As we indicated at (1.3), we have the following fact, proven in the Appendix.

Fact 1. *The function $f(t) = t^{-1}(1 - e^t)\log(1 - e^{-t})$ is a strictly decreasing bijection from $\mathbb{R}^+$ onto $\mathbb{R}^+$.*

Returning to (2.7), one solution $(x,y) \in \mathcal{V}$ to (2.6) has $x = f^{-1}(s/r) > 0$, where f is the function in Fact 1. Using symmetric calculations and $x = -\log(1 - e^{-y})$ on $\mathcal{V}$, we also find that this solution has $y = f^{-1}(r/s) > 0$.

Thus, for fixed $(r,s) \in S_{\varepsilon_0}$, we have identified the promised point $(x,y) \in \mathcal{V}$ by $x = f^{-1}(s/r)$ and $y = f^{-1}(r/s)$. It remains to show that this fixed point $(x,y) \in \mathcal{V}$ satisfies Conditions (i)–(iii) of Theorem 2.1.

Verifying Condition (i). Consider $\mathbf{dir}(x,y) = \mathrm{span}_{\mathbb{C}}\{(xH_x, yH_y)\} \overset{(2.5)}{=} \mathrm{span}_{\mathbb{C}}\{(-xe^{-x}, -ye^{-y})\}$. Since $x = f^{-1}(s/r)$ and $y = f^{-1}(r/s)$ solve (2.6), write

$$k = r\frac{e^x}{x} = s\frac{e^y}{y} \quad \Longrightarrow \quad r = kxe^{-x} \text{ and } s = kye^{-y} \quad \Longrightarrow \quad (r,s) \in \mathbf{dir}(x,y),$$

as promised. To see that (x,y) is strictly minimal, we will apply the following lemma (which we prove in the next subsection).

Lemma 2.1. *If $(a,b) \in \mathcal{V} \cap (0,1)^2$ (cf. (2.5)), then (a,b) is strictly minimal on $\mathcal{V}$.*

To apply Lemma 2.1, we have already noted that $x, y > 0$ (cf. Fact 1), so it remains to show that $x, y < 1$. Now, since $(r,s) \in S_{\varepsilon_0}$ (cf. (1.2)), we have $\frac{s}{r}, \frac{r}{s} > \varepsilon_0$, where we note from (1.2) that $f(1) = \varepsilon_0$. Thus, since f is monotone decreasing, we must have $x = f^{-1}(s/r) < 1$ and $y = f^{-1}(r/s) < 1$ so that, by Lemma 2.1, (x,y) is strictly minimal.

Verifying Condition (ii). It is easy to verify Condition (ii). Indeed, by the Inverse Function Theorem, $(x,y) \in \mathcal{V}$ varies smoothly as $(r,s) \in S_{\varepsilon_0}$ varies. Moreover, $(x,y) \in \mathcal{V}$ varies over the trace of the curve $(f^{-1}(t), f^{-1}(1/t))$ for $t \in [\varepsilon_0, 1/\varepsilon_0]$ (see Figure 1), which is the continuous image of a compact set.

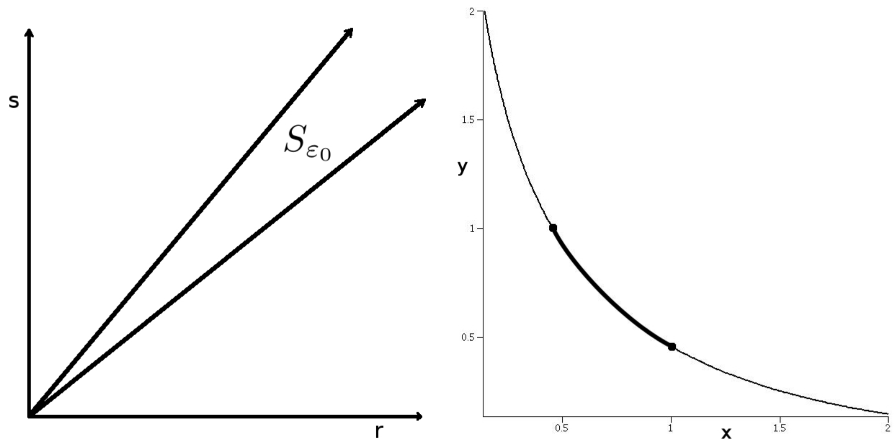

FIGURE 1. The Sector S_{ε_0} in the rs-plane (left), and the real section of $\mathcal{V}$ (right) in the xy-plane with the set of minimal points (x, y) corresponding to S_{ε_0} plotted in bold.

Verifying Condition (iii). It is slightly tedious to verify Condition (iii). To begin, note that the function $G(x, y) = e^{-y}$ from (2.5) is never zero. We show that the function $Q(x, y)$ is always positive on $\mathcal{V} \cap (\mathbb{R}^+ \times \mathbb{R}^+)$. It is easy to check that

$$\begin{aligned} Q(x, y) &= -y^2 H_y^2 x H_x - y H_y x^2 H_x^2 - x^2 y^2 (H_y^2 H_{xx} + H_x^2 H_{yy} - 2 H_x H_y H_{xy}) \\ &= xye^{-x-y} \left[ye^{-y} + xe^{-x} - xy(e^{-y} + e^{-x}) \right], \end{aligned}$$

which we defined in (1.4). Now, $x > 0$ and $y > 0$ imply that $xye^{-x-y} > 0$, so we will disregard this factor. Moreover, in the expression above, $e^{-y} + e^{-x} = 1$ on $\mathcal{V}$, so we consider the simpler function

$$P(x, y) = ye^{-y} + xe^{-x} - xy,$$

and prove that $P(x, y)$ is positive on $\mathcal{V} \cap (\mathbb{R}^+ \times \mathbb{R}^+)$. Using $1 + x < e^x$ on $\mathcal{V}$, we have

$$\begin{aligned} 1 + x < e^x = \frac{1}{1 - e^{-y}} \quad \implies \quad x < \frac{e^{-y}}{1 - e^{-y}} \quad \implies \\ P(x, y) = ye^{-y} - x\left(y - e^{-x}\right) > ye^{-y} - \left(\frac{e^{-y}}{1 - e^{-y}}\right)\left(y - e^{-x}\right) = \frac{e^{-y}}{1 - e^{-y}}\left(e^{-x} - ye^{-y}\right). \end{aligned}$$

Clearly, the first factor above is positive on $y > 0$, so it suffices to consider $R(x, y) = e^{-x} - ye^{-y}$. However, on $\mathcal{V}$, we have $R(x, y) = 1 - e^{-y}(1 + y) > 0$, since $1 + y < e^y$. This confirms Condition (iii).

We now conclude the proof of Theorem 1.3. We apply Theorem 2.1 to $e^{-y}/(e^{-x} + e^{-y} - 1)$ (cf. (2.2), (2.5)) to conclude that for $r, s \to \infty$ with $(r, s) \in S_{\varepsilon_0}$,

$$A_{r,s} = \left(e^{-y} + O(s^{-1/2})\right) \frac{1}{\sqrt{2\pi}} x^{-r} y^{-s} \sqrt{\frac{ye^y}{sQ(x, y)}}$$

and so, by (2.3), we have

$$[b^{r-1} a^s] = r!s! A_{r,s} = r!s! \left(e^{-y} + O(s^{-1/2})\right) \frac{1}{\sqrt{2\pi}} x^{-r} y^{-s} \sqrt{\frac{ye^y}{sQ(x, y)}},$$

as promised.

2.3. **Proof of Lemma 2.1.** Fix $(a, b) \in \mathcal{V} \cap (0,1)^2$, where we recall from (2.5) that

$$\mathcal{V} = \left\{(x,y) \in \mathbb{C}^2 : H(x,y) = e^{-x} + e^{-y} - 1 = 0\right\}.$$

To show that (a, b) is stictly minimal on $\mathcal{V}$, we prove that

$$\min_{z \in \overline{D}(0,a)} \Re e\, e^{-z} \text{ is achieved at only } z = a, \tag{2.8}$$

and so symmetrically,

$$\min_{w \in \overline{D}(0,b)} \Re e\, e^{-w} \text{ is achieved at only } w = b.$$

As such, if $(z, w) \in \overline{D}(0, a) \times \overline{D}(0, b)$ satisfies $(z, w) \neq (a, b)$, we have

$$\Re e\, H(z,w) = \Re e\, e^{-z} + \Re e\, e^{-w} - 1 > e^{-a} + e^{-b} - 1 = 0,$$

in which case $H(z, w) \neq 0$ and so $(z, w) \notin \mathcal{V}$.

To show (2.8), we solve for all $z \in \overline{D}(0, a)$ which minimize $\Re e\, e^{-z}$ (and show that only $z = a$ works). We begin by making the following initial considerations. For $z \in \overline{D}(0, a)$, we write $z = u + iv$ so that $u^2 + v^2 \leq a^2$ and $\Re e\, e^{-z} = e^{-u} \cos v$. Note that $\Re e\, e^{-z} = e^{-u} \cos v$ is a harmonic function (it is the real part of an analytic function). As such, the maximum principle [1, Sec. 6.2] ensures that the minimum value of $\Re e\, e^{-z} = e^{-u} \cos v$ over $\overline{D}(0, a)$ is attained at the boundary $\partial D(0, a) = \{z \in \overline{D}(0, a) : |z|^2 = u^2 + v^2 = a^2\}$. As such, we will always assume $v \neq 0$, for otherwise $z = u = \pm a$, and the following hold:

(1) $z = -a$ does not minimize $\Re e\, e^{-z} = e^{-u} \cos v$ (since $e^a > e^{-a}$ with $a > 0$),
(2) $z = a$ is what we promise in (2.8).

To minimize $\Re e\, e^{-z} = e^{-u} \cos v$ subject to $|z|^2 = u^2 + v^2 = a^2$, we use the Lagrange multiplier rule: for a scalar $\lambda \in \mathbb{R}$, set

$$\nabla\left(e^{-u}\cos v\right) = \lambda \nabla\left(u^2+v^2\right) \quad \Longrightarrow \quad \left(-e^{-u}\cos v, -e^{-u}\sin v\right) = (2\lambda u, 2\lambda v). \tag{2.9}$$

In light of (2.9), we may now also assume $u \neq 0$, for otherwise, $\cos v = 0$ implies $0 \neq v = k\pi/2$ for some (necessarily nonzero) $k \in \mathbb{Z}$, and so $1 > a = v^2 = k^2\pi^2/4 \geq \pi^2/4 > 2$. At this stage of our analysis, we will have proven (2.8) if we can show that

$$\text{no } z = u + iv \in \partial\overline{D}(0,a) \text{ with } u \neq 0 \neq v \text{ will solve (2.9)}. \tag{2.10}$$

Indeed, with $u \neq 0 \neq v$, we may rewrite (2.9) to say

$$\frac{\sin v}{v} = -2\lambda e^{-u} = \frac{\cos v}{u} \quad \Longrightarrow \quad v = u \tan v, \tag{2.11}$$

which we now rewrite in polar coordinates. For $\theta \in [0, 2\pi)$, let $u = a\cos\theta$ and $v = a\sin\theta$, where we may assume from $u \neq 0 \neq v$ that $\theta \notin \{0, \pi/2, \pi, 3\pi/2\}$. Then (2.11) is equivalent to

$$a\sin\theta = a\cos\theta\tan(a\sin\theta) \quad \Longrightarrow \quad \tan\theta = \tan(a\sin\theta) \quad \Longrightarrow \quad a\sin\theta = \theta + k\pi \text{ where } k \in \mathbb{Z}. \tag{2.12}$$

We now investigate the possible values of $k \in \mathbb{Z}$ in (2.12). Since $a \in (0, 1)$ and $\theta \in (0, 2\pi)$, we have

$$|\theta + k\pi| = a|\sin\theta| < |\sin\theta| < \min\{1, \theta\} \tag{2.13}$$

and so only $k \in \{-1, -2\}$ are possible. However, $k \neq -1$ because the two sides of $a\sin\theta = \theta - \pi$ will have opposite signs. Moreover, $k \neq -2$ because

$$2\pi - \theta = -a\sin\theta = a\sin(-\theta) = a\sin(2\pi-\theta) \leq a|\sin(2\pi-\theta)| < |\sin(2\pi-\theta)| \leq 2\pi - \theta.$$

Thus, we have proven (2.10), and hence (2.8), which concludes the proof of Lemma 2.1.

3. Proof of Theorem 1.2

The proof of Theorem 1.2 involves specializing Theorem 1.3 to the case $r - 1 = \lfloor (n-1)/2 \rfloor$ and $s = \lceil (n-1)/2 \rceil$, and therefore, consists of calculations. We split the proof into two cases, depending on the parity of n, and begin with the easier (cleaner) case.

3.1. **Proof of Theorem 1.2 (n is even).** When n is even, we have $r = s = n/2$, and so $x = y = f^{-1}(1)$, where f is the function defined in Fact 1. As we saw in the proof of Theorem 1.3, (x, y) is a strictly minimal point on $\mathcal{V} = \{(x, y) \in \mathbb{C}^2 : e^{-x} + e^{-y} - 1 = 0\}$, and so we easily calculate $x = y = \log 2$. It remains to substitute $r = s = n/2$ and $x = y = \log 2$ into the asymptotic expression of Theorem 1.2. We proceed in a piecemeal way.

To begin, note first that with $x = y = \log 2$ and $r = s = n/2$, we have

$$Q(x,y) = \frac{1}{4}(1-\log 2)\log^3 2 \quad \Longrightarrow \quad x^{-r}y^{-s}\sqrt{\frac{ye^{-y}}{sQ(x,y)}} = \frac{2}{\log^{n+1} 2} \cdot \frac{1}{\sqrt{n(1-\log 2)}}. \tag{3.1}$$

With $r = s = n/2$, we use Stirling's approximation to conclude

$$\frac{1}{\sqrt{2\pi}} r!s! = \frac{1}{\sqrt{2\pi}}\left(\left(\frac{n}{2}\right)!\right)^2 = \left(1 + O\left(\frac{1}{n}\right)\right)\frac{\sqrt{n}}{2^{n+1}} n!. \tag{3.2}$$

With $y = \log 2$ and $s = n/2$, the error term of Theorem 1.3 is

$$e^{-y} + O(s^{-1/2}) = \frac{1}{2} + O\left(\frac{1}{\sqrt{n}}\right). \tag{3.3}$$

Multiplying (3.1)–(3.3) together yields

$$\left[b^{\lfloor (n-1)/2 \rfloor} a^{\lceil (n-1)/2 \rceil}\right] = \left(\frac{1}{2\log 2\sqrt{(1-\log 2)}} + O\left(\frac{1}{\sqrt{n}}\right)\right)\left(\frac{1}{2\log 2}\right)^n n!,$$

which is slightly stronger than promised.

3.2. **Proof of Theorem 1.2 (n is odd).** The calculations here are similar, and we claim that (3.1)–(3.3) still hold as long as we dampen them to include some error factor of $(1 + o(1))$, where $o(1) \to 0$ as $n \to \infty$. When n is odd, we have $r = (n + 1)/2 = s + 1$, and it is routine to update (3.2) to say

$$\frac{1}{\sqrt{2\pi}} r!s! = \frac{1}{\sqrt{2\pi}}\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)! = (1 + o(1))\frac{\sqrt{n}}{2^{n+1}} n!. \tag{3.4}$$

Now,

$$x = f^{-1}\left(\frac{s}{r}\right) = f^{-1}\left(1 - \frac{2}{n+1}\right) \quad \text{and} \quad y = f^{-1}\left(\frac{r}{s}\right) = f^{-1}\left(1 + \frac{2}{n-1}\right), \tag{3.5}$$

and so by the continuity of f^{-1}, we have $x, y \to f^{-1}(1) = \log 2$ as $n \to \infty$. By continuity alone, we may update (3.3) and parts of (3.1) to say

$$e^{-y} + O(s^{-1/2}) = \frac{1}{2} + o(1) \tag{3.6}$$

and

$$Q(x,y) = (1 + o(1))\frac{1}{4}(1-\log 2)\log^3 2 \quad \Longrightarrow \quad \sqrt{\frac{ye^{-y}}{sQ(x,y)}} = (1 + o(1))\frac{2}{\log 2} \cdot \frac{1}{\sqrt{n(1-\log 2)}}. \tag{3.7}$$

With its large exponents, we need to be slightly more careful with the the factor $x^{-r}y^{-s}$ in (3.1).

The function f in Fact 1 is analytic, and Fact 1 guarantees that it is invertible and that f' is nonzero. As such, f^{-1} is analytic, and so we use Taylor's theorem to perform a linear approximation of $f^{-1}(z)$ near $a = 1$. Set $d = (f^{-1})'(1)$. We update (3.5) to say

$$\begin{aligned} x = f^{-1}\left(1 - \frac{2}{n+1}\right) &= f^{-1}(1) - d\frac{2}{n+1} + \Theta\left(\frac{1}{n^2}\right) = \log 2 - d\frac{2}{n+1} + \Theta\left(\frac{1}{n^2}\right) \\ \text{and} \quad y = f^{-1}\left(1 + \frac{2}{n-1}\right) &= f^{-1}(1) + d\frac{2}{n-1} + \Theta\left(\frac{1}{n^2}\right) = \log 2 + d\frac{2}{n-1} + \Theta\left(\frac{1}{n^2}\right), \end{aligned} \tag{3.8}$$

and so $xy = \log^2 2 + \Theta(1/n^2)$. Now, with $r = (n+1)/2 = s+1$,

$$x^r y^s = x(xy)^s = x\left(\log^2 2 + \Theta\left(\frac{1}{n^2}\right)\right)^s \overset{(3.8)}{=} \left(\log^{2s+1} 2\right)\left(1+\Theta\left(\frac{1}{n}\right)\right)\left(1+\Theta\left(\frac{1}{n^2}\right)\right)^{\frac{n-1}{2}}$$
$$= (1+o(1))\log^n 2 \quad \Longrightarrow \quad x^{-r}y^{-s} = \frac{1}{\log^n 2}(1+o(1)). \quad (3.9)$$

Now, multiplying (3.4), (3.6), (3.7), and (3.9) yields

$$\left[b^{\lfloor (n-1)/2\rfloor} a^{\lceil (n-1)/2\rceil}\right] = \left(\frac{1}{2\log 2\sqrt{(1-\log 2)}} + o(1)\right)\left(\frac{1}{2\log 2}\right)^n n!,$$

as promised.

4. Proof of Lemma 1.6

We indicated that Lemma 1.6 is (mostly) a consequence of a result of Clarke, Steingrímsson, and Zeng [5]. To state that result, define, for a permutation $\pi \in S_n$, $\mathrm{DesBot}(\pi) = \{\pi(i) : \pi(i) < \pi(i-1)\}$ to be the (so-called) *Descent Bottoms Set*.

Lemma 4.1 (Clarke, Steingrímsson, Zeng [5])**.** *There exists a bijection $\Phi : S_n \to S_n$ so that for any $\pi \in S_n$, $\mathrm{Exc}(\pi) = \mathrm{DesBot}(\Phi(\pi))$.*

At the end of this section, we include (for completeness) a proof of Lemma 4.1 using a different[1] bijection Φ (which was suggested to us by Emeric Deutsch [8]).

4.1. **Proof of Lemma 1.6: Statement (1).** To prove Statement (1) of Lemma 1.6 from Lemma 4.1, we make the following two observations.

Observation 1. *Let $\pi \in S_{n+1}$ have no fixed points. Then, π admits no stretching pairs iff there exists $\ell \in [n]$ so that $\mathrm{Exc}(\pi) = [\ell]$.*

Proof. Indeed, clearly no $\pi \in S_{n+1}$ whose excedence set takes the form $\mathrm{Exc}(\pi) = [\ell]$ can have stretching pairs. Suppose $\pi \in S_{n+1}$ has neither stretching pairs nor fixed points, and let $\ell \in [n]$ be the maximum integer for which $[\ell] \subseteq \mathrm{Exc}(\pi)$. Then $[\ell] = \mathrm{Exc}(\pi)$, since otherwise $j \in \mathrm{Exc}(\pi) \setminus [\ell]$ would give the stretching pair $\ell + 1 < j$. □

Observation 2. *There is a bijection $F : C_{n+1} \to S_n$ so that for each $\pi \in C_{n+1}$ and for each $\ell \in [n]$, $\mathrm{Exc}(\pi) = [\ell]$ iff $\mathrm{DesBot}(F(\pi)) = [\ell - 1]$.*

Proof. Indeed, fix $\pi \in C_{n+1}$ and define $F(\pi) = p \in S_n$ by the rule $p(i) = \pi^{n+1-i}(1) - 1$. For $j \in [n]$, we show $j+1 \in \mathrm{Exc}(\pi)$ iff $j \in \mathrm{DesBot}(p)$. Write $j = p(i)$ for some $i \in [n]$ so that $j+1 = p(i)+1 = \pi^{n+1-i}(1)$ and $\pi(j+1) = \pi^{n+1-(i-1)}(1) = p(i-1)+1$. Now, $\pi(j+1) - (j+1) = p(i-1) - p(i)$ is positive if, and only if, $j+1 \in \mathrm{Exc}(\pi)$ and if, and only if, $j = p(i) \in \mathrm{DesBot}(p)$. □

To prove Statement (1) of Lemma 1.6, we note that every $\pi \in C_{n+1}$ is without fixed points, and so Observation 1 gives the disjoint union

$$C^*_{n+1} = \bigcup_{\ell=1}^{n} \{\pi \in C_{n+1} : \mathrm{Exc}(\pi) = [\ell]\} \quad \Longrightarrow \quad \left|C^*_{n+1}\right| = \sum_{\ell=1}^{n} \left|\, \{\pi \in C_{n+1} : \mathrm{Exc}(\pi) = [\ell]\} \,\right|.$$

Applying Observation 2 and Lemma 4.1 (in this order) then gives

$$\left|C^*_{n+1}\right| = \sum_{k=0}^{n-1} \left|\, \{\pi \in S_n : \mathrm{DesBot}(\pi) = [k]\} \,\right| = \sum_{k=0}^{n-1} \left|\, \{\pi \in S_n : \mathrm{Exc}(\pi) = [k]\} \,\right| = \sum_{k=0}^{n-1} [b^k a^{n-1-k}],$$

as promised.

---

[1] According to Ehrenborg and Steingrímsson [9], there are several variations of the bijection Φ, many of which seem to originate in early work of Foata and Schützenberger [12].

4.2. **Proof of Lemma 1.6: Statement (2).** Let $A_n = A_n(\{21{-}34, 34{-}21\})$ denote the set of permutations $\pi \in S_n$ which avoid both $21{-}34$ and $34{-}21$ as generalized patterns, and write $\alpha_n = \alpha_n(\{21{-}34, 34{-}21\}) = |A_n|$. For an $(n+1)$-cycle $\pi \in C_{n+1}$, we say that a stretching pair $\pi(i) < i < j < \pi(j)$ is *typical* if $\pi(j) \neq n+1$, and *exceptional* otherwise. Let $E_{n+1} \subset C_{n+1}$ denote the set of $(n+1)$-cycles π whose only stretching pairs are exceptional. The following statement[2] is an easy fact from Section 4.1 of [6].

Proposition 4.2 (Cooper et al. [6]). *There is a bijection $\phi : C_{n+1} \to S_n$ with the property that $\pi \in C^*_{n+1} \cup E_{n+1}$ iff $\phi(\pi) \in A_n$.*

Proposition 4.2 gives the identity $\alpha_n = |E_{n+1}| + |C^*_{n+1}|$, and hence, $\alpha_n \geq |C^*_{n+1}|$, which is the lower bound in Statement (2) of Lemma 1.6. It remains to prove the upper bound. We shall establish an injection $\iota : E_{n+1} \to C^*_{n+2}$, in which case $|E_{n+1}| \leq |C^*_{n+2}|$ and the upper bound in Statement (2) of Lemma 1.6 then follows.

To define ι, it will be convenient to work rather with $C_{n+2}[0, n+1]$, which is the set of $(n+2)$-cycles defined on the elements $[0, n+1] = \{0, 1, \ldots, n+1\}$. We take $C^*_{n+2}[0, n+1]$ to be the set of $\pi \in C_{n+2}[0, n+1]$ with no stretching pairs. Now, fix $\pi \in E_{n+1}$, and write $j_0 = \pi^{-1}(n+1)$. By construction, all stretching pairs $i < j$ in $\pi \in E_{n+1}$ have $j = j_0$. Define $p = \iota(\pi) \in C^*_{n+2}[0, n+1]$ by the following rule: $p(0) = n+1$, $p(j_0) = 0$, and $p(i) = \pi(i)$ for all remaining $i \in [n+1]$. Clearly, ι is an injection and $p = \iota(\pi)$ is an $(n+2)$-cycle on the elements $[0, n+1]$. Note that all stretching pairs of π have been eliminated in $p = \iota(\pi)$, and no new ones arise.

4.3. **Proof of Lemma 4.1.** Define the bijection Φ as follows. Let $\pi \in S_n$ be given in its *standard cycle decomposition*, $SCD(\pi)$, i.e., each cycle begins with its smallest entry and cycles appear in ascending order according to their initial entries. Define $p = \Phi(\pi) \in S_n$ to be the permutation which, in one-line notation $p = (p_1, \ldots, p_n)$ where $i \mapsto p_i$, is obtained by first reversing the order of entries within each cycle of π, and then removing all parentheses. For example, if $\pi = (1\ 5\ 2\ 8)(3\ 6\ 7)(4\ 9)$, then $\Phi(\pi) = (8, 2, 5, 1, 7, 6, 3, 9, 4)$.

It is easy to show that Φ is a bijection by constructing its inverse function. Fix $p = (p_1, \ldots, p_n) \in S_n$, and construct $SCD(\pi)$ for $\pi = \Phi^{-1}(p)$ as follows. Find the index k_1 for which $p_{k_1} = 1$. Take the consecutively indexed entries from p_1 to p_{k_1} and reverse their order to obtain the first cycle for π. Find the index $k_2 > k_1$ for which p_{k_2} is the smallest element of $[n] \setminus \{p_1, \ldots, p_{k_1}\}$. Construct the second cycle for π by reversing the order of the entries starting with p_{k_1+1} and ending with p_{k_2}. Having constructed the first j cycles for π, find the index $k_{j+1} > k_j$ so that $p_{k_{j+1}}$ is the smallest element of $[n] \setminus \{p_1, p_2, \ldots, p_{k_j}\}$, and obtain the next cycle by reversing the order of the entries starting with p_{k_j+1} and ending with $p_{k_{j+1}}$. By construction, π will be in standard cycle decomposition and $\Phi(\pi) = p$.

We now verify that Φ has the promised property: for all $\pi \in S_n$, $\mathrm{Exc}(\pi) = \mathrm{DesBot}(p = \Phi(\pi))$. To that end, fix $\pi = (p_1, \ldots, p_n) \in S_n$. For $j \in [n]$, we show $j \in \mathrm{Exc}(\pi) \iff j = \mathrm{DesBot}(p)$. We consider two cases, depending on how j appears in $SCD(\pi)$.

Case 1. *In $SCD(\pi)$, j does not appear last in its cycle.* Then, $\pi(j)$ is the next consecutive entry appearing in the same cycle as j. Set $k = p^{-1}(j)$ so that $p_k = j$. Then, by the construction of Φ, $p_{k-1} = \pi(j)$. Now,

$$j \in \mathrm{Exc}(\pi) \quad \iff \quad p_{k-1} = \pi(j) > j = p_k \quad \iff \quad j = p_k \in \mathrm{DesBot}(p),$$

as desired.

Case 2. *In $SCD(\pi)$, j appears last in its cycle.* Then, $\pi(j)$ is the first entry in the cycle containing j. Since π is in standard cycle decomposition, $\pi(j) \leq j$, in which case $j \notin \mathrm{Exc}(\pi)$. If j is in the first cycle of π, then $p_1 = j \notin \mathrm{DesBot}(p)$, as desired. Assume, therefore, that j is not in the first cycle of π.

[2] The bijection ϕ in Proposition 4.2 is easy to state. If $(n+1\ a_1\ \ldots\ a_n) \in C_{n+1}$ is given in cyclic notation $n+1 \mapsto a_1 \mapsto \ldots \mapsto a_n \mapsto n+1$, then $\phi(\pi) = (a_1, \ldots, a_n) \in S_n$, which we've written in customary notation $i \mapsto a_i$.

Let the cycle immediately preceding that of j begin with a, and let the cycle containing j begin with b. Then, $a < b \leq j$. Moreover, $p_{k-1} = a < j = p_k$, so that $p_k = j \notin \mathrm{DesBot}(p)$, as desired.

Appendix: Proof of Fact 1

Recall that we wish to prove $f(t) = t^{-1}(1-e^t)\log(1-e^{-t})$ is a strictly decreasing bijection from $\mathbb{R}^+$ onto $\mathbb{R}^+$. To that end, it is straightforward to check that $\lim_{t\to 0^+} f(t) = +\infty$ and $\lim_{t\to+\infty} f(t) = 0$ (the numerator of the second limit tends to 1). By continuity, we immediately conclude that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is onto.

The remainder of the Appendix is reserved to proving that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly decreasing, and in particular, that f' is negative on $\mathbb{R}^+$. In what follows, we make the substitution $t = \log p$, where $p \in (1, \infty)$, and we consider the function

$$F(p) = f(\log p) = \frac{(1-p)\log(1-\frac{1}{p})}{\log p} = \frac{(1-p)\,(\log(p-1)-\log p)}{\log p} = \frac{(1-p)\log(p-1)}{\log p} - (1-p).$$

We will show that F' is negative on $(1,\infty)$, and since $t = \log p$ is increasing in $p \in (1,\infty)$, we will infer that f' is negative on $\mathbb{R}^+$. This will conclude our proof.

Observe that

$$F'(p) - 1 = \frac{1}{\log^2 p}\left[\log p\left[(-1)\log(p-1) + (1-p)\frac{1}{p-1}\right] - (1-p)\log(1-p)\cdot\frac{1}{p}\right],$$

so that

$$F'(p) = -\frac{p(\log p)\log(p-1) + p\log p + (1-p)\log(p-1) - p\log^2 p}{p\log^2 p}.$$

We show that the numerator

$$N(p) = p(\log p)\log(p-1) + p\log p + (1-p)\log(p-1) - p\log^2 p$$

is positive on $(1,\infty)$, which then implies that F' is negative on $(1,\infty)$. For that, we first note that $N(p) \to 0^+$ as $p \to 1^+$. As such, it suffices to show that N' is positive on $(1,\infty)$.

Observe that

$$\begin{aligned} N'(p) &= (\log p)\log(p-1) + \log(p-1) + \frac{p}{p-1}\log p + \log p + 1 - \log(p-1) - 1 - \log^2 p - 2\log p \\ &= (\log p)\left[\frac{1}{p-1} - \log\left(1 + \frac{1}{p-1}\right)\right]. \end{aligned}$$

Note that $\log p > 0$ on $(1,\infty)$. The second factor above is also positive, since $\log(1+x) < x$ holds for all $x = 1/(p-1) \in \mathbb{R}$.

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